Laplace's deterministic paradise lost

Miguel Angel Fuentes
Santa Fe Institute, USA
Instituto Balseiro and CONICET, Argentina

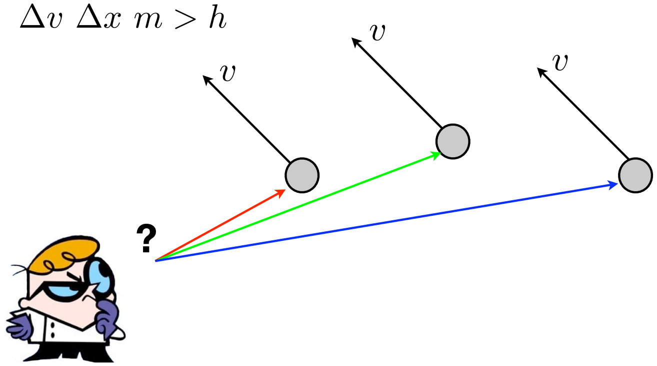
"We may regard the **present state** of the universe as **the effect of its past and the cause of its future**. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace **in a single formula** the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect *nothing would be uncertain and the future just like the past would be present before its eyes."*

Pierre-Simon Laplace, ~1800

Comments: Heisenberg uncertainty principle

In the framework of quantum mechanics -one of the most successful created theories- is not possible to know -at the same time- the position and velocity of a given object. (~1927)

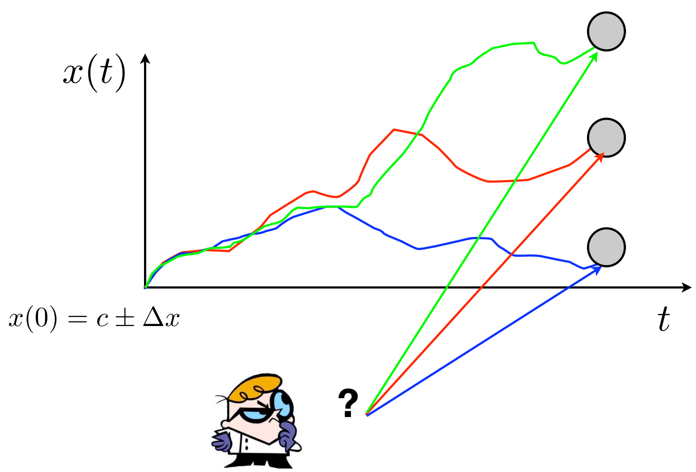
$$\Delta v \; \Delta x \; m > h$$

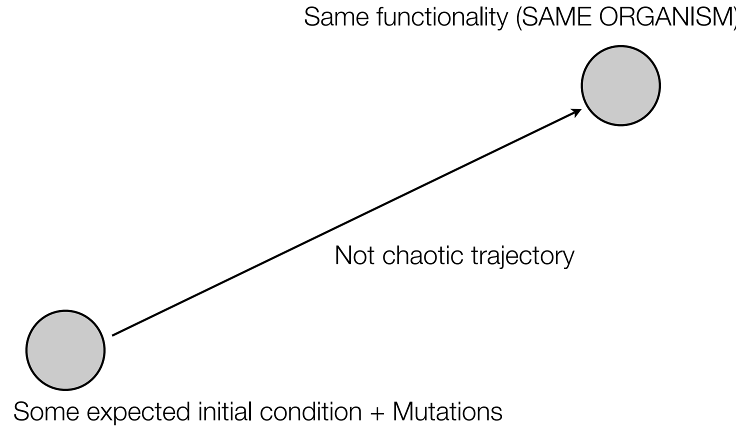


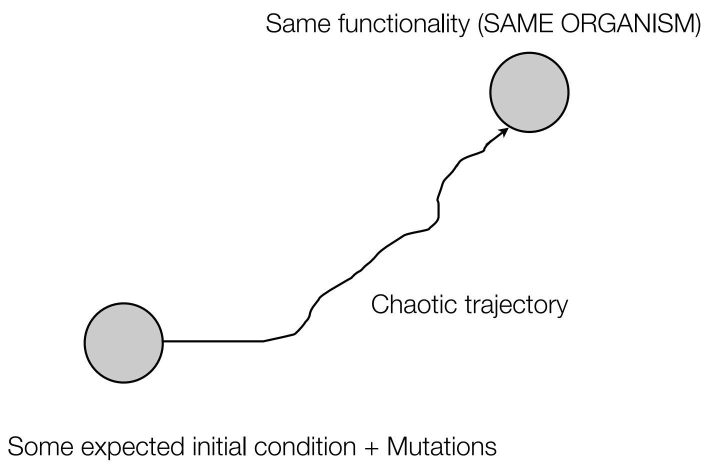
Einstein was very unhappy about this apparent randomness in nature. His views were summed up in his famous phrase, **'God does not play dice.'**

(from Stephen Hawking's lecture)

In chaotic (deterministic) systems



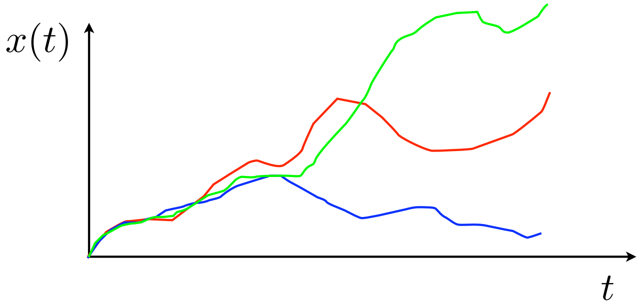




Many process can happen between regular-chaotic behavior

$$|x_{green}(t) - x_{blue}(t)| \sim e^{\lambda t}$$

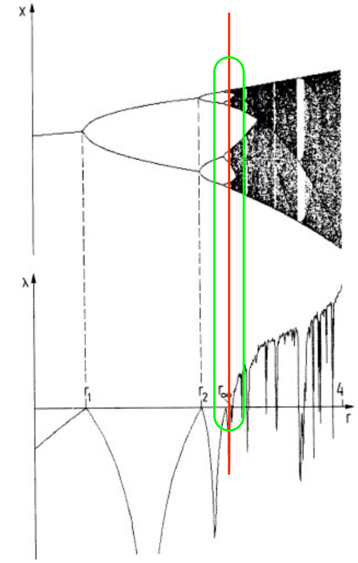
λ : Lyapunov exponent, positive for chaos



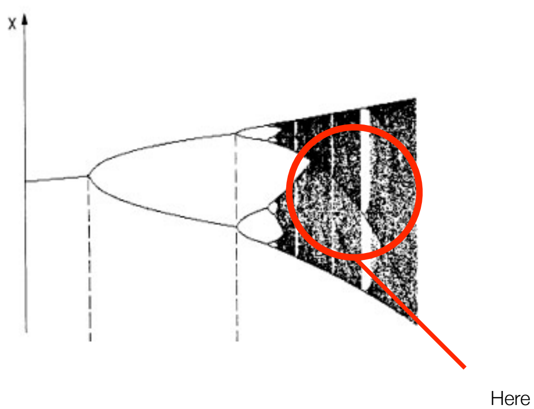
Some process are neither chaotic nor regular

accumulation points

Liapunov exponent



There are important theorems for random variables that are valid for chaotic systems



Until now we were discussing:

Chaos and uncertainty

Its connection with the idea of random processes

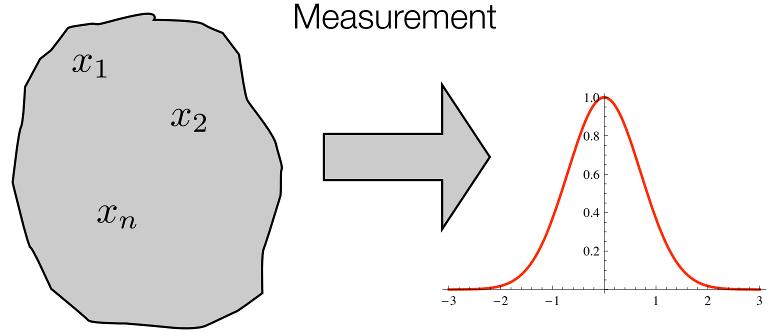
Chaos and Lyapunov exponent

I mention that some important theorem for random variables are valid for deterministic variables too. Let move to one of them (**VERY important one**.)

Abraham de Moivre (1667) a french mathematician famous for de Moivre's formula (**friend of** Isaac Newton, Edmund Halley, and James Stirling, nice crowd!)

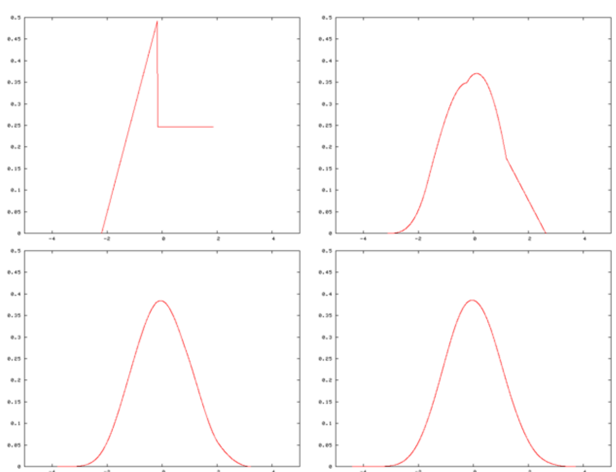
$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

The central limit theorem states that the **sum of a large number** of independent and identically-distributed **random variables** will be approximately normally distributed (i.e., following a **Gaussian distribution**) if the random variables have a finite variance (~*1773*).



Example: $z = \frac{\sum_1^N x_i}{\sqrt{N}}$

$P_N(z)$



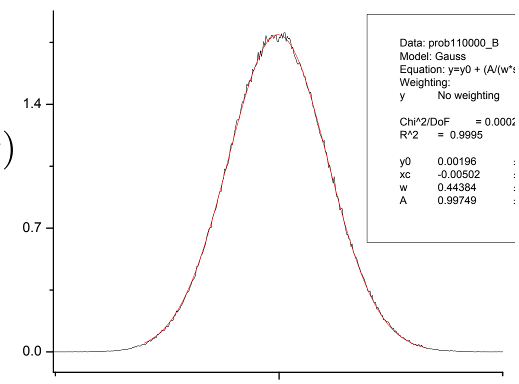
z

For the logistic map at positive Lyapunov exponent (CHAOS)

$$x_{i+1} = T(x_i) = 1 - a x_i^2$$

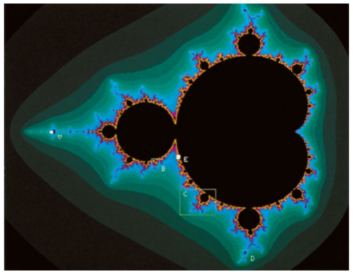
$$y := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} f(x_i)$$

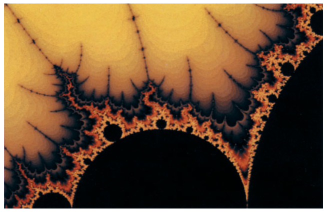
$P(y)$

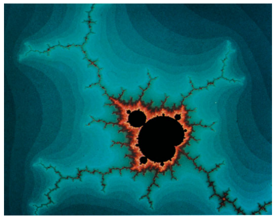


y

Interlude: fractals



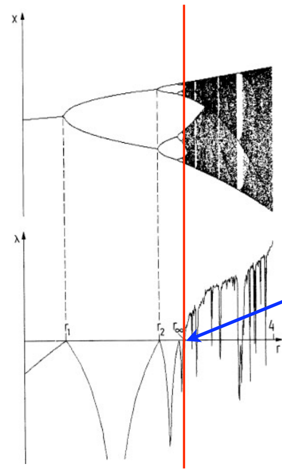




The set M is called "Mandelbrot's set" after B. B. Mandelbrot who first published (1980) a picture of M (see Fig. 94). It shows that M has also a fractal structure (but it is no Julia set). This study was extended by Peitgen and Richter (1984). If c does not belong to M, then $\lim_{n\to\infty} f_c^n(0) \to \infty$. Therefore, they define "level curves" in the following way: color a starting point according to the number of iterations it needs to leave a disk with a given radius R. As shown by Douady and Hubbard (1982), lines of equal color can be interpreted as equipotential lines if the set M is considered to be a charged conductor. Plates VIII–XV show the fascinating results of this procedure which brings us back to Ruelles' remark at the beginning of this section.

From "Deterministic Chaos", Schuster & Just

In chaos the central limit distribution theorem holds



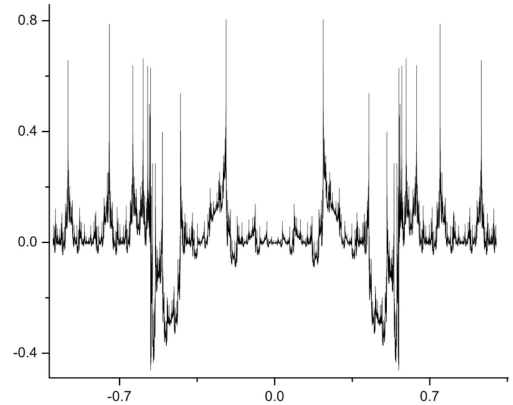
What happens at the edge of chaos? (zero Lyapunov exponent)

There are long (some times unconstructive and boring) debates on what mechanical statistical framework is appropriated to use at the so called: edge -or onset- of chaos. (For me,) more clear and exact is to use, in this case, a less fancy expression: **"at zero Lyapunov exponent."**

At zero Lyapunov exponent ?

$$y := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} f(x_i)$$

y (Sum)

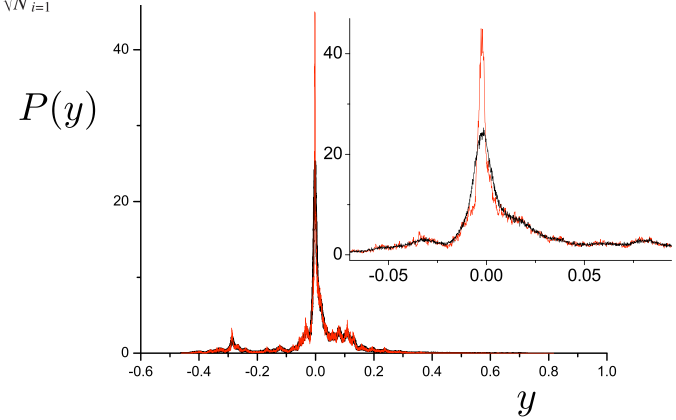


x_0 (initial condition)

Self-similarity behavior

Limit distribution **at zero Lyapunov exponent**
(and near zero Lyap. exponent)

$$y := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} f(x_i)$$



Things we have discussed:

Chaos and uncertainty

Deterministic variables behave as random variables

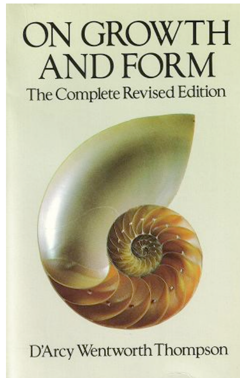
Simple systems are amazingly rich and there are still important open questions in its behavior

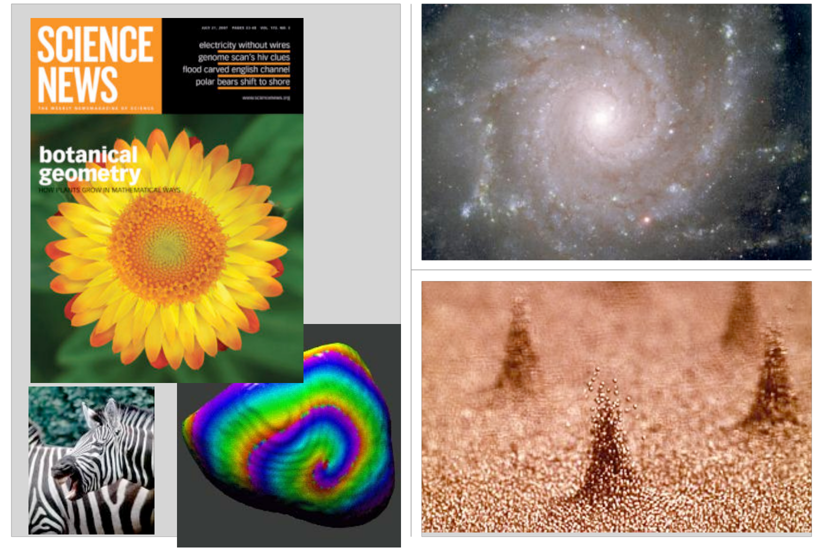
Patterns

Miguel Angel Fuentes
Santa Fe Institute, USA
Instituto Balseiro and CONICET, Argentina

Themes

- Patterns
- Tools for pattern descriptions
- Reaction diffusion equation
- Waves and excitable media
- Non local mechanism
- Beyond standard analysis





PDE, Reaction Diffusion Models:

"*Under certain conditions spatially inhomogeneous patterns can evolve by diffusion driven instability*" A. M. Turing, 1952

The chemical basis of morphogenesis, A. M. Turing, Phil. Trans. Roy. Soc. London, B237, 37-72, 1952.

Minimal model

$$\begin{aligned} \partial_t u &= f(u,v) + d_u \nabla^2 u \\ \partial_t v &= g(u,v) + d_v \nabla^2 v \end{aligned}$$

Linear stability analysis

(u_0, v_0): homogenous steady state, i. e.:

$$f(u_0, v_0) = g(u_0, v_0) = 0$$

$$\mathbf{w} = \begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix}$$

$$\dot{\mathbf{w}} = \mathbf{A}\mathbf{w} \qquad \mathbf{A} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$$

$$\mathbf{w} \sim e^{\lambda t} \quad \begin{cases} \lambda > 0 & \mathbf{w} \to \infty \\ \lambda < 0 & \mathbf{w} \to 0 \end{cases}$$

Conditions for stability of the homogeneous state

The eigenvalues are given by the solution of

$$\begin{vmatrix} f_u - \lambda & f_v \\ g_u & g_v - \lambda \end{vmatrix} = 0$$

Then there are two solution: λ_1, λ_2 from the equation:

$$\lambda^2 - (f_u + g_v)\lambda + (f_u g_v - f_v g_u) = 0$$

Linear stability, $\Re[\lambda] < 0$,is guaranteed if

$$tr\mathbf{A} = f_u + g_v < 0, \quad |\mathbf{A}| = f_u g_v - f_v g_u > 0$$

$$\mathbf{A} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$$

Conditions for instability of the homogeneous state

$$\begin{aligned} \partial_t u &= f(u,v) + d_u \nabla^2 u \\ \partial_t v &= g(u,v) + d_v \nabla^2 v \end{aligned}$$

Linear version

$$\partial_t \mathbf{w} = \mathbf{A}\mathbf{w} + \mathbf{D}\nabla^2 \mathbf{w} \qquad \mathbf{w} = \sum_k c_k e^{\lambda t} \mathbf{W}_k(\mathbf{r})$$

Using cos functions

$$\begin{aligned} \lambda \mathbf{W}_k &= \mathbf{A}\mathbf{W}_k + \mathbf{D}\nabla^2 \mathbf{W}_k \\ &= \mathbf{A}\mathbf{W}_k - \mathbf{D}k^2 \mathbf{W}_k \end{aligned}$$

Eigenvalues equation

$$|\lambda \mathbf{I} - \mathbf{A} + \mathbf{D}k^2| = 0 \qquad \mathbf{w} = \sum_k c_k e^{\lambda t} \mathbf{W}_k(\mathbf{r})$$

$$\lambda^2 + \lambda[k^2(1+d) - (f_u + g_v)] + h(k^2) = 0$$

$$h(k^2) = dk^4 (df_u + g_v)k^2 + |\mathbf{A}|$$

Linear stability, $\Re[\lambda] > 0$,is guaranteed if

$$df_u + g_v > 0, \qquad (df_u + g_v)^2 - 4d|\mathbf{A}| > 0$$

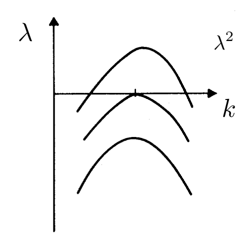
Conditions

Stability of the homogeneous state

$$tr\mathbf{A} = f_u + g_v < 0, \quad |\mathbf{A}| = f_u g_v - f_v g_u > 0$$

Instability of some modes

$$df_u + g_v > 0, \qquad (df_u + g_v)^2 - 4d|\mathbf{A}| > 0$$



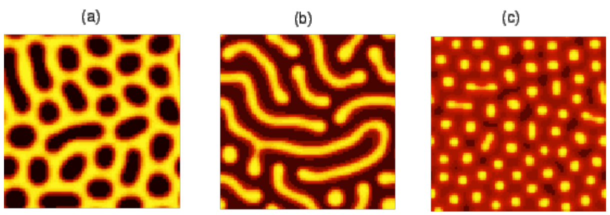
$$\lambda^2 + \lambda[k^2(1+d) - (f_u + g_v)] + h(k^2) = 0$$

Discussion

Easy frame work

Two diffusion coefficient

Extensive literature



Waves

Minimal non linear model with diffusion

$$\partial_t u = ku(1-u) + D\partial_{xx} u$$ Fisher, 1937

$$\partial_t u = u(1-u) + \partial_{xx} u$$

Static frame

$$u(x,t) = U(z), \qquad z = x - ct$$

$$U'' + cU' + U(1-U) = 0$$

$$\lim_{z\to\infty} U(z) = 0, \qquad \lim_{z\to-\infty} = 1$$

In the plane (U,V)

$$U' = V, \qquad V' = -cV - U(1-U)$$

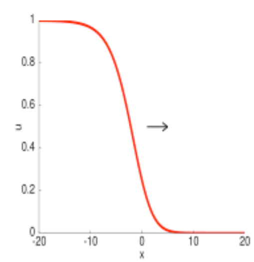
Linear stability analysis

$$(0,0): \qquad \lambda_\pm = [-c \pm (c^2-4)^{1/2}]/2$$

stable node if $c^2 \geq 4$

stable spiral if $c^2 < 4$

$$(1,0): \qquad \lambda_\pm = [-c \pm (c^2+4)^{1/2}]/2$$



$$c \geq c_{min} = 2\sqrt{kD}$$

Discussion

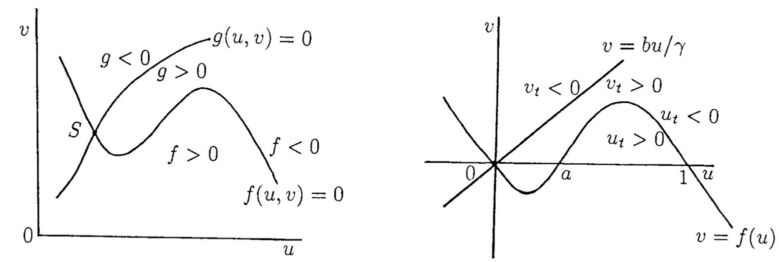
Finite domain: localized structure

Critical Size

Excitable media

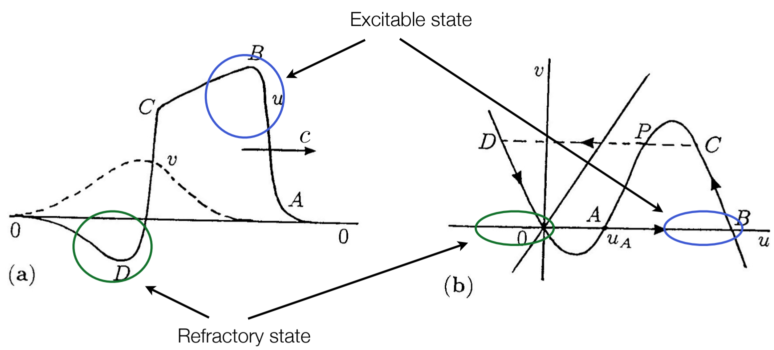
$$u_t = f(u,v), \qquad v_t = g(u,v)$$

Nullclines

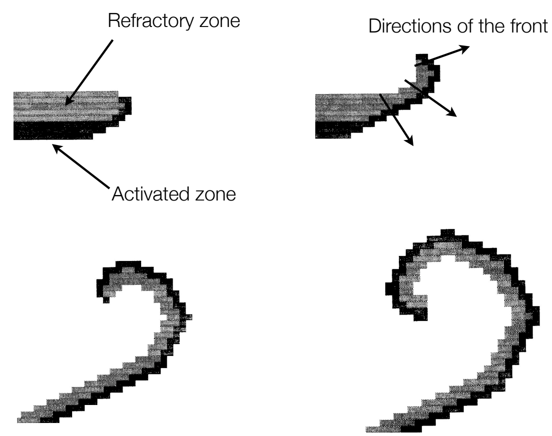


Excitable media with space

$$\partial_t u = f(u,v) + \partial_{xx} u \qquad \partial_t v = g(u,v)$$



Spiral waves:
self sustained spatiotemporal structures



A. S. Mikailov, Foundation of Synergetics

More dimensions

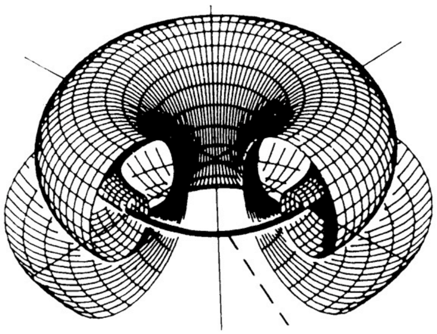


Fig. 3.33. Scroll ring

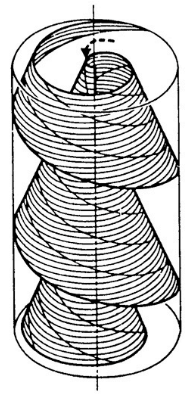


Fig. 3.34. Straight twisted scroll

A. S. Mikailov, Foundation of Synergetics

Discussion

Analytical method for velocity, geometrical properties of the spiral, etc.

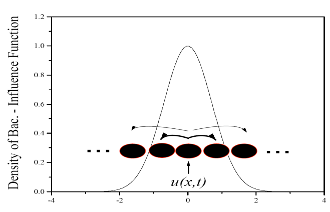
Multi-arms spirals

Interaction of spirals

Cellular automata methods

Non local mechanisms

$$\partial_t u = f(u) + g(u) \int_\Omega G(x,x')\ u(x')\ dx', \qquad \int_\Omega G = 1$$



$$\partial_t u = f(u) + g(u) \int_\Omega G(x,x')\ u(x')\ dx', \qquad \int_\Omega G = 1$$

Near the stable point

$$\partial_t u = f'(u_0)u + g'(u_0)u_0 u + g(u_0) \int G\ u\ dx', \qquad u_0 \text{ is a stable point}$$

Doing the Fourier transform, i. e. $u \sim e^{ikx+\lambda t}$

$$\lambda = f'(u_0) + g'(u_0)u_0 + g(u_0)F(k), \qquad \text{with} \;\; F(k) = \int G e^{iky} dy$$

Equation with space variables

$$\partial_t u = f(u) + g(u) \int_\Omega G(x,x')\; u(x')\; dx' + d\nabla^2 u$$

Condition for unstable modes

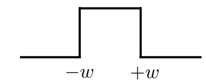
$$F(k) > \frac{Dk^2 - f'(u_0) - g'(u_0)u_0}{g(u_0)}$$

Example

$$\partial_t u = f(u) + g(u) \int_\Omega G(x,x')\; u(x')\; dx' + d\nabla^2 u$$

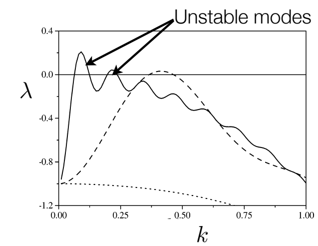
$$f(u) = au, \qquad g(u) = -bu$$

$$G(x,x') = \frac{\theta[w-(x-x')]\theta[w+(x-x')]}{2w}$$



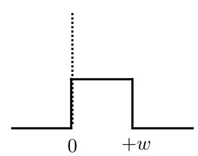
Note: $G \longrightarrow \delta$, is the Fisher equation

$$\lambda = -a\frac{\sin(kw)}{kw} - dk^2$$



Traveling waves - Non local Interactions

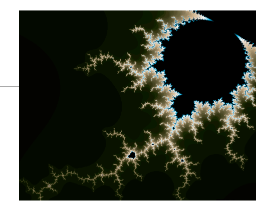
$$G(x,x') = \frac{\theta[x]\theta[w+(x-x')]}{w}$$



$$\Re[F(k)] \;=\; \frac{sin(kw)}{kw} \qquad \longleftarrow \text{Patters}$$

$$\Im[F(k)] \;=\; \frac{1}{kw} - \frac{sin(kw)}{kw} \neq 0 \qquad \longleftarrow \text{Oscillations}$$

Frontiers



1- Lacking of first principles descriptions

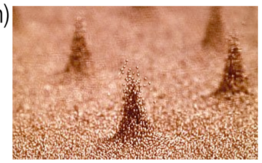
2- A different geometry

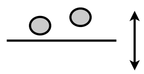
Lacking of first principles descriptions

Example

Non continuos Systems (hoping for an approach)

Granular system

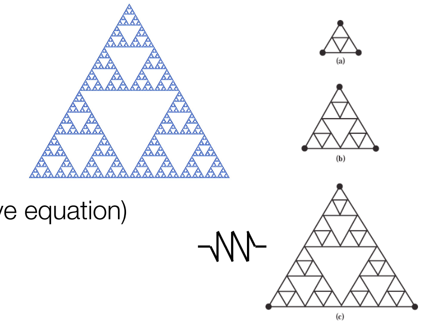




Roughly speaking

- Markovian
- Non Markovian, memory

A different geometry



Example

The Sierpinski gasket (and its wave equation)

Non continuos operator

$$\begin{cases} \partial_t^2 u = \nabla^2 u \\ \partial_t^2 u = \lim_{m\to\infty} \frac{3}{2} 5^m H_m u, \quad H_m u = O(x,y) \end{cases}$$

Yamaguti et al., Translations of mathematical monographs, American Mathematical Society, 1997

Different solutions (eigenfunctions)

Not only different patterns, but physics etc

Using a similar approach is my hope that in networks studies where we can define a geometry -distances, etc- it will be possible to define a : "*Calculus on network*"

Conclusions

- Linear stability analysis is a fundamental tool
- Turing patterns has many application, we must have a critic view
- In the literature: studies for ‘amplitude equations’
- Non continuos systems need special attention
- Dynamical systems knowledge is also necessary
- The geometry where the system lives is very important
- Calculus on network