Introduction to Nonlinear Dynamics

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Chaos:

Complex behavior, arising in a deterministic nonlinear dynamic system, which exhibits two special properties:

- sensitive dependence on initial conditions
- characteristic structure…

Systems that exhibit chaos are ubiquitous; many of them are also simple, well-known, and “well-understood”
Where chaos turns up:

• Flows (of fluids, heat, …)
  - Eddy in creek
  - Weather
  - Vortices around marine invertebrates
  - Air/fuel flow in combustion chambers
Where chaos turns up:

• Driven nonlinear oscillators
  - Pendula
  - Hearts
  - Fireflies

- and lots of other electronic, chemical, & biological systems
Where chaos turns up:

- Classical mechanics
  - three-body problem
  - paired black holes
  - pulsar emission
  - ….
- Protein folding
- Population biology
- And many, many other fields (including yours)

Hut & Bahcall
• continuous time systems:
  • time proceeds smoothly
  • “flows”
  • modeling tool: differential equations

• discrete time systems:
  • time proceeds in clicks
  • “maps”
  • modeling tool: difference equation
A useful graphical solution technique:

- “cobweb” diagram
- return map
- correlation plot
Bifurcations

*Qualitative* changes in the dynamics caused by changes in *parameters*
Bifurcations

Qualitative changes in the dynamics caused by changes in *parameters*:

- Heart: pathology
- Eddy in creek: water level
- Olfactory bulb: smell
- Brain: blood chemicals
- etc. etc.
Bifurcations in the logistic map:

Discrete time: should not connect dots!!
These plots stolen from Strogatz
• chaos

• veils/bands: places where chaotic attractor is dense (UPOs)
• chaos

• veils/bands: places where chaotic attractor is dense (UPOs)

• *period-doubling cascade* @ low $R$
Feigenbaum number
Universality!

Feigenbaum number and many other interesting chaotic/dynamical properties hold for any 1D map with a quadratic maximum.

Proof: renormalizations. See Strogatz §10.7

Don’t take this too far, though…
• chaos

• veils/bands: places where chaotic attractor is dense (UPOs)

• period-doubling cascade @ low R

• *windows of order within the chaos, complete with their own period-doubling cascades* (e.g., 3 to 6 to 12)
A bit more lore on periods and chaos:

- Sarkovskii (1964)
- Yorke (1975)
- Metropolis *et al.* (1973)
• chaos
• veils/bands: places where chaotic attractor is dense (UPOs)
• period-doubling cascade @ low R
• windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)
• small copies of object embedded in it (fractal)
Fractals and Chaos…

The connection: *many (most)* chaotic systems have fractal state-space structure.
Fractals

- non-integer Hausdorff dimension
- self-similar

Images from Gleick.

Examples: Cantor set, coastlines, trees, lungs, clouds, drainage basins, …
In computer graphics…

Matthew Ward, WPI
http://davis.wpi.edu/~matt/courses/fractals/trees.html
In maps:

Newton’s method on $x^4 - 1 = 0$
That was all about maps.

- discrete time systems:
  - time proceeds in clicks
  - “maps”
  - modeling tool: difference equation
Next: *flows*.

- continuous time systems:
  - time proceeds smoothly
  - “flows”
  - modeling tool: differential equations
But first…

original piece

chaotic mapping

chaotic variation