Introduction to Nonlinear Dynamics

Santa Fe Institute

Complex Systems Summer School

4-6 June 2012

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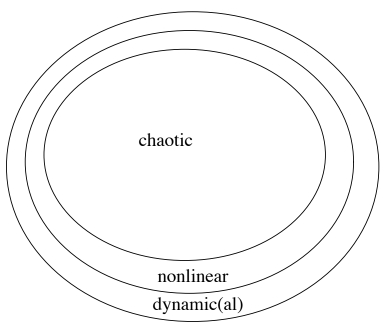
Chaos

Complex behavior, arising in a deterministic nonlinear dynamic system, which exhibits two special properties:

- sensitive dependence on initial conditions
- characteristic structure…



Chaos

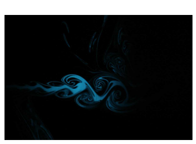
Complex behavior, arising in a deterministic nonlinear dynamic system, which exhibits two special properties:

- sensitive dependence on initial conditions
- characteristic structure…

Systems that exhibit chaos are ubiquitous; many of them are also simple, well-known, and "well-understood"

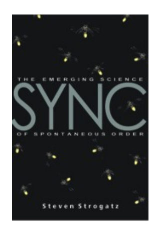
Where nonlinear dynamics turns up

- Flows (of fluids, heat, …)
  - Eddy in creek
  - Weather
  - Vortices around marine invertebrates
  - Air/fuel flow in combustion chambers



Where nonlinear dynamics turns up

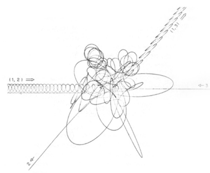
- Driven nonlinear oscillators
  - Pendula
  - Hearts
  - Fireflies



  - and lots of other electronic, chemical, & biological systems

Where nonlinear dynamics turns up

- Classical mechanics
  - three-body problem
  - paired black holes
  - pulsar emission
  - ….
- Protein folding
- Population biology
- And many, many other fields (including yours)



Hut & Bahcall *Ap.J.* ***268****:319*

- continuous time systems:
  - time proceeds smoothly
  - "flows"
  - modeling tool: differential equations
- discrete time systems:
  - time proceeds in clicks
  - "maps"
  - modeling tool: difference equation

A useful graphical solution technique

- "cobweb" diagram
- *aka* return map
- *aka* correlation plot

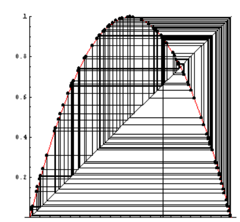


Image from Doug Ravenel's website at URochester

Bifurcations

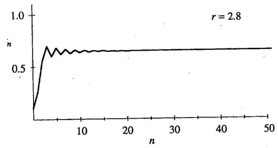
Qualitative changes in the dynamics caused by changes in *parameters*

Bifurcations

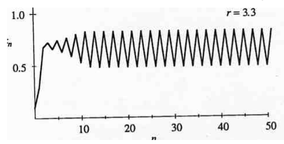
Qualitative changes in the dynamics caused by changes in *parameters:*

- Heart: pathology
- Eddy in creek: water level
- Olfactory bulb: smell
- Brain: blood chemicals
- etc. etc.

Bifurcations in the logistic map



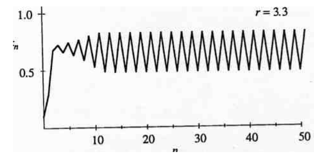
R=2.8



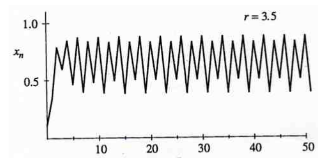
R=3.3

Note: in discrete time plots, <u>it makes no sense to connect dots!!</u>

Plots from *Strogatz*

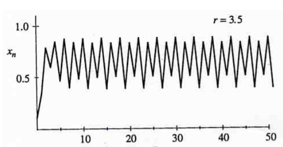


R=3.3

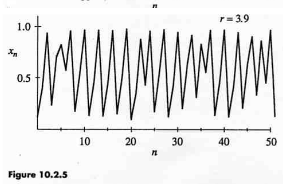


R=3.5

Plots from *Strogatz*



R=3.5



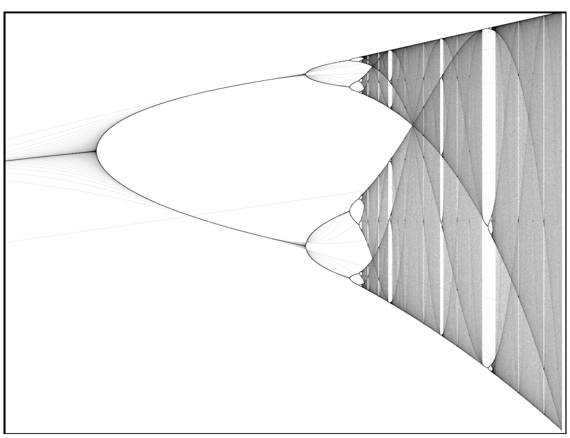
R=3.9

Plots from *Strogatz*

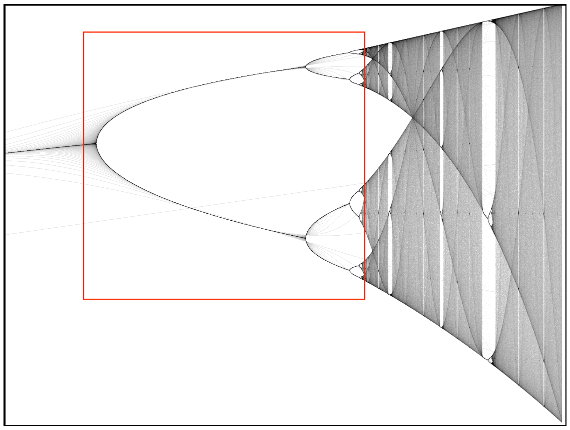





Courtesy of Allison Brown;
Chaos, in preparation.

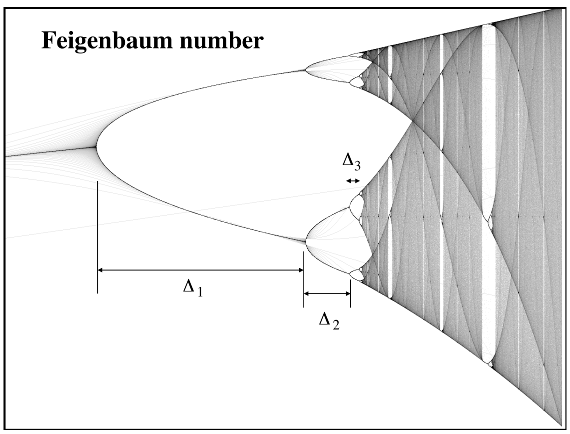


- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)



- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- *period-doubling cascade @ low R*

Feigenbaum number

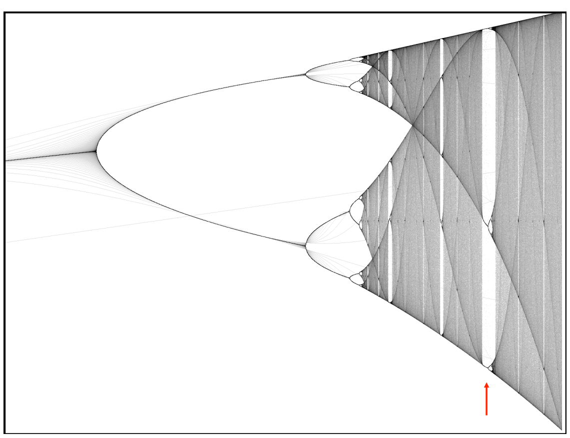


Universality!

Feigenbaum number and many other interesting chaotic/dynamical properties hold for any 1D map with a quadratic maximum.

Proof: renormalizations. See Strogatz §10.7

Don't take this too far, though…



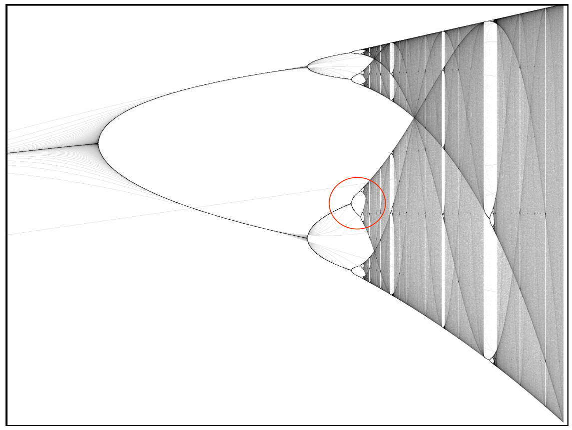
- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- period-doubling cascade @ low R
- *windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)*

A bit more lore on periods and chaos

- Sarkovskii (1964)
  3, 5, 7, …3x2, 5x2, …$3x2^2$, $5x2^2$, … 2^2, 2, 1

Yorke (1975)

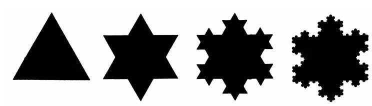
- Metropolis *et al.* (1973)

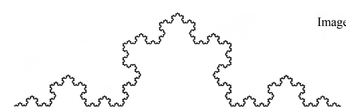


- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- period-doubling cascade @ low R
- windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)
- *small copies of object embedded in it (fractal)*

Fractals

- non-integer Hausdorff dimension
- self-similar

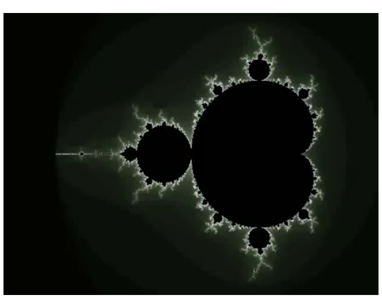




Images from Gleick

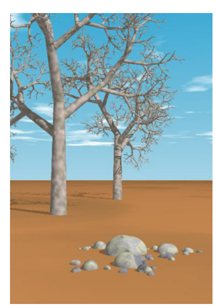
Examples: Cantor set, coastlines, trees, lungs, clouds, drainage basins, …

The Mandelbrot set



www.youtube.com/watch?v=G_GBwuYuOOs

Fractals in computer graphics



Matthew Ward, WPI
davis.wpi.edu/~matt/courses/fractals/trees.html

Fractals in maps

Newton's method on $x^4 - 1 = 0$

From Strogatz

Fractals and chaos…

The connection: *many (most)* chaotic systems have fractal state-space structure.

But **not** "*all*."

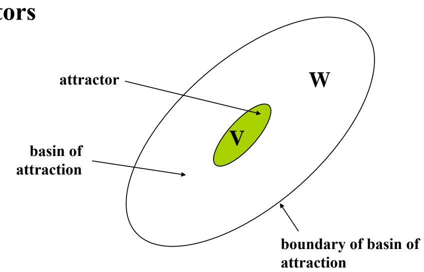
So far: mostly about *maps.*

- discrete time systems:
  - time proceeds in clicks
  - "maps"
  - modeling tool: difference equation

Next up: *flows*

- continuous time systems:
  - time proceeds smoothly
  - "flows"
  - modeling tool: differential equations

Attractors



- Attractors exist only in dissipative systems!
- Dissipation ⟺ contraction of state space under the influence of the dynamics
- Can still have chaos if no dissipation…just not chaotic *attractors*

Conditions for chaos in continuous-time systems

Necessary:

- Nonlinear
- At least three state-space dimensions (NB: only one needed in maps)

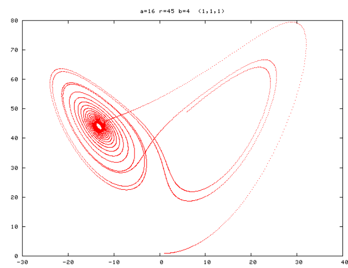
Necessary and sufficient:

- "Nonintegrable"

  i.e., cannot be solved in closed form

Concepts: review

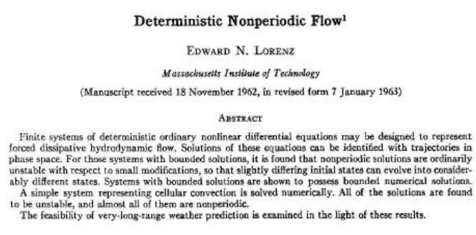
- State variable
- State space
- Initial condition
- Trajectory
- Attractor
- Basin of attraction
- Transient
- Fixed point (un/stable)
- Bifurcation
- Parameter



A Lorenz applet:

www.exploratorium.edu/complexity/java/lorenz.html

(Note: by Jim Crutchfield, who will be here next week)



Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

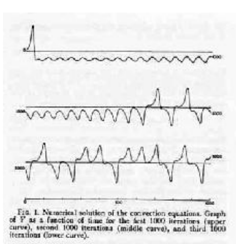
ABSTRACT

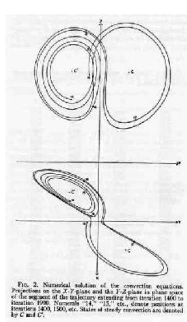
Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

J. Atm. Sci. **20**:130



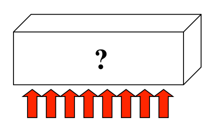


- Equations:

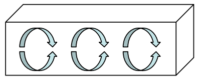
$x' = a(y-x)$

$y' = rx - y - xz$

$z' = xy - bz$

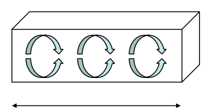


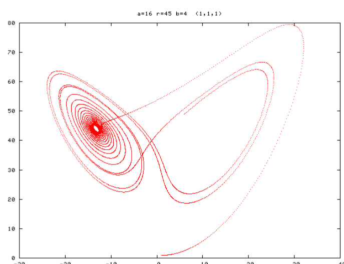
(first three terms of a Fourier expansion of the Navier-Stokes eqns)



- State variables:
  - *x* convective intensity
  - *y* temperature
  - *z* deviation from linearity in the vertical convection profile

- Parameters:
  - *a* Prandtl number - fluids property
  - *r* Rayleigh number - related to ΔT 
  - *b* aspect ratio of the fluid sheet

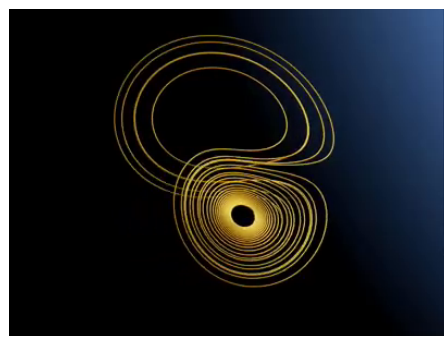


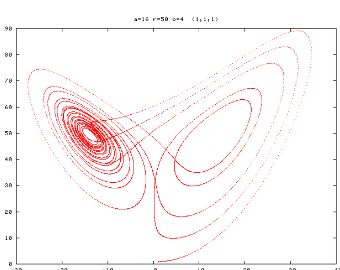


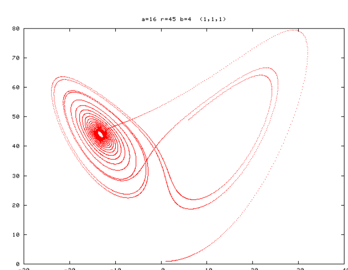
$x' = 16(y-x)$

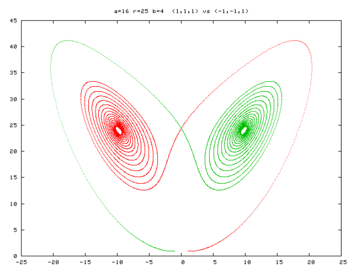
$y' = 45x - y - xz$

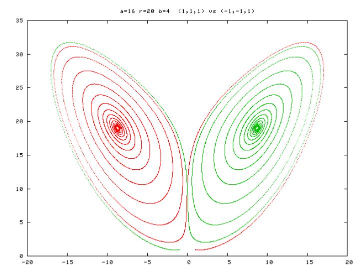
$z' = xy - 4z$

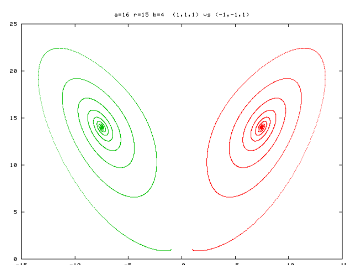




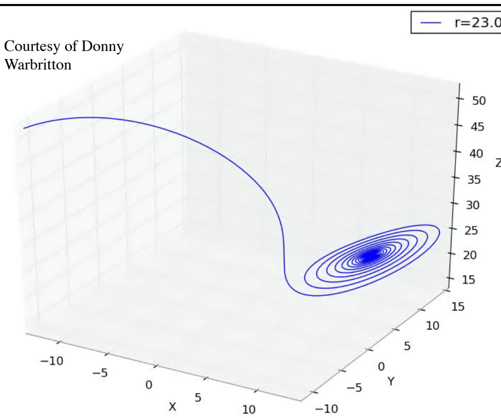








Courtesy of Donny Warbritton



Attractors

Four types:

- fixed points
- limit cycles (*aka* periodic orbits)
- quasiperiodic orbits
- chaotic attractors

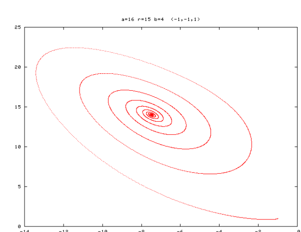
A nonlinear system can have any number of attractors, of all types, sprinkled around its state space

Their basins of attraction (plus the basin boundaries) *partition* the state space

And there's no way, *a priori*, to know where they are, how many there are, what types, etc.

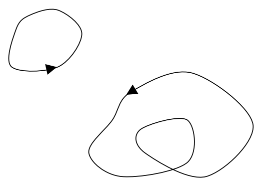
Attractors

- Fixed point



Attractors

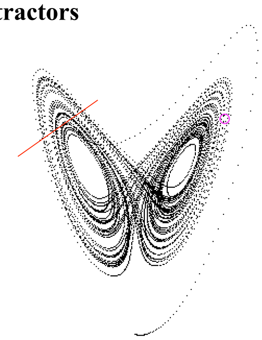
- Limit cycle



Attractors

- Quasi-periodic orbit…

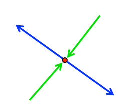
"Strange" or chaotic attractors



- *often* fractal
- covered densely by trajectories
- exponential divergence of neighboring trajectories…

Lyapunov exponents

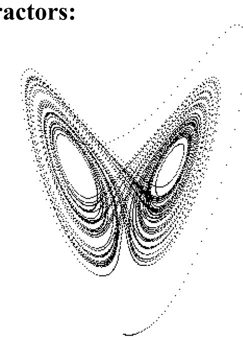
- nonlinear analogs of eigenvalues: one λ for each dimension



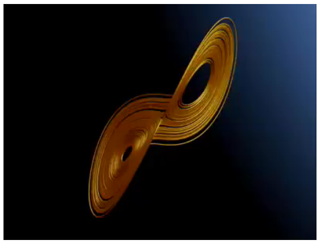
Lyapunov exponents: summary

- nonlinear analogs of eigenvalues: one λ for each dimension
- negative λ_i compress state space; positive λ_i stretch it
- $\Sigma\lambda_i < 0$ for dissipative systems
- long-term average in definition; biggest one dominates as t → infinity
- *positive λ is a signature of chaos*
- λ_i are same for all ICs in one basin

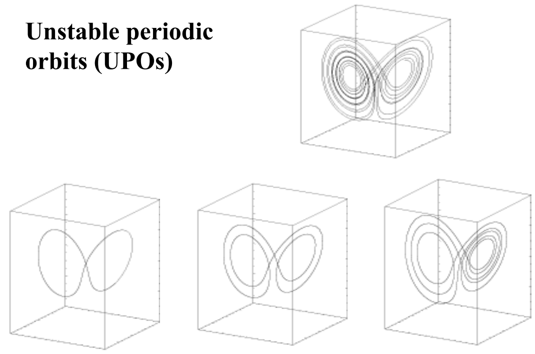
"Strange" or chaotic attractors:



- exponential divergence of neighboring trajectories
- *often* fractal
- covered densely by trajectories
- contain an infinite number of "unstable periodic orbits"…

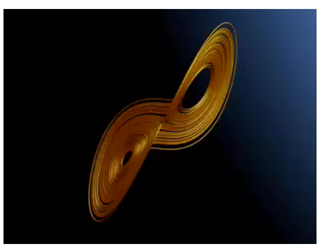


Unstable periodic orbits (UPOs)

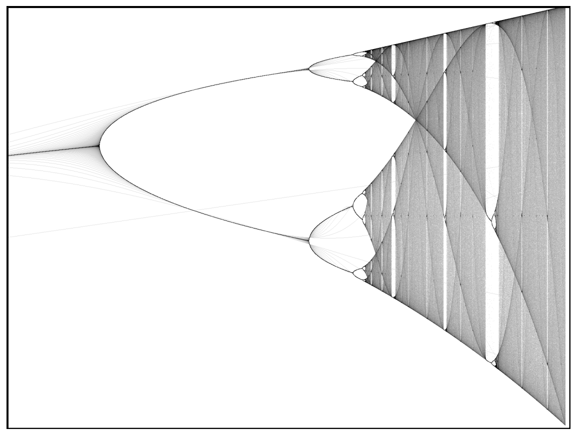


Bradley/Mantilla, *Chaos* **12**:596

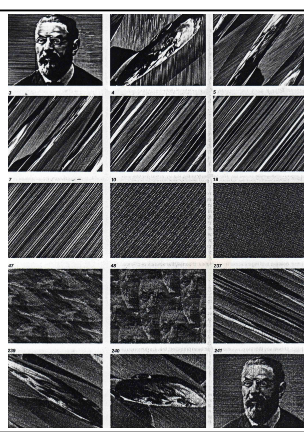
Attractor "bones"…



© 2006 Jos Leys and Etienne Ghys; www.josleys.com



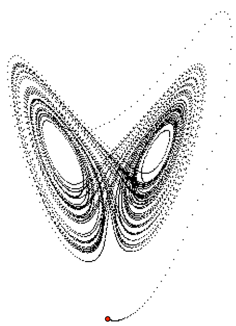
Poincare recurrence

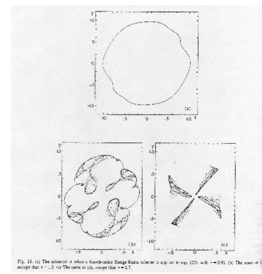


Crutchfield *et al*.
Chaos **255**:46

The rest of today…

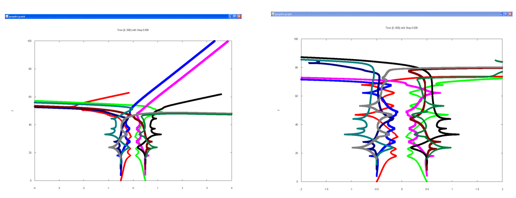
- Lunch (cafeteria downstairs)
- Dynamics Lab I: (here)
  - Meet here at 1:30pm
  - Bring your laptop, if you have one here
  - Lab handouts on the CSSS wiki
- 3pm — Intro to student projects (here)
- 4:15pm — Start thinking & talking about those projects!





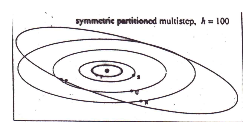
Different timestep

Lorenz, *Physica D* **35**:229

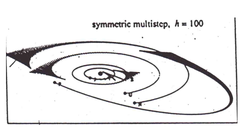


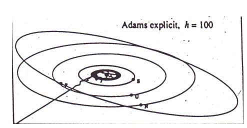
Different arithmetic

N. Ross Ph.D. thesis, Ucolorado, 2008



Different solver algorithm…





Moral: numerical methods can run amok in "interesting" ways…

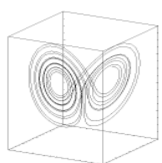
- can cause distortions, bifurcations, etc.
- and these look a lot like *real, physical* dynamics…
- source: algorithms, arithmetic system, timestep, etc.
- Q: what could you do to diagnose whether your results included spurious numerical dynamics?

Moral: numerical methods can run amok in "interesting" ways…

- can cause distortions, bifurcations, etc.
- and these look a lot like *real, physical* dynamics…
- source: algorithms, arithmetic system, timestep, etc.
- Q: what could you do to diagnose whether your results included spurious numerical dynamics?
    - *change the timestep*
    - *change the method*
    - *change the arithmetic*

So ODE solvers make mistakes.

…and chaotic systems are sensitively dependent on initial conditions….



…??!?

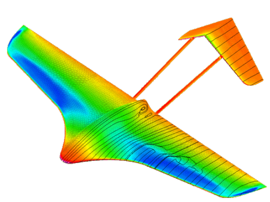
Shadowing lemma

Every* noise-added trajectory on a chaotic attractor is *shadowed* by a true trajectory.

Important: this is for *state* noise, not *parameter* noise.

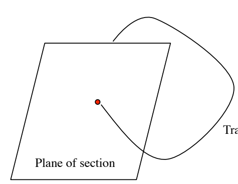
(*) Caveat: not if the noise bumps the trajectory out of the basin

Solving *PDEs*

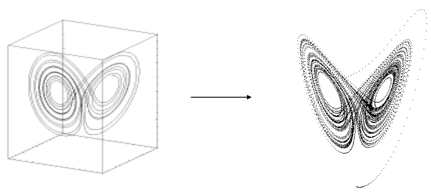


www.tecplot.com

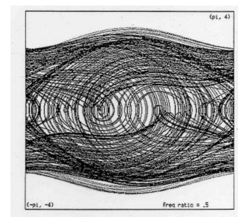
Section

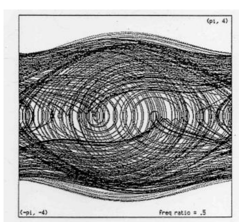


Not the same thing as a projection!

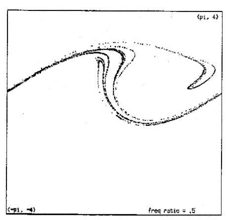


The driven damped pendulum



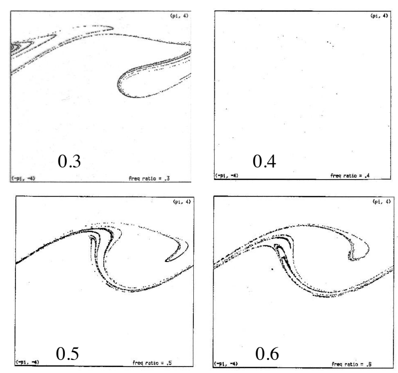


trajectory

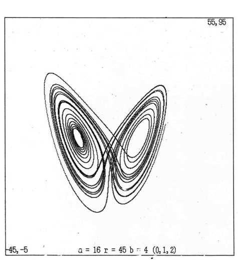


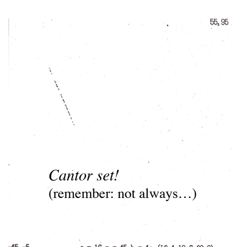
Poincare section

What bifurcations look like on a Poincare section

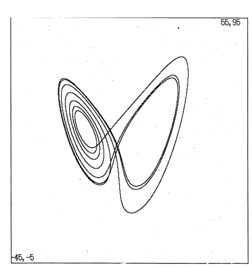


The Lorenz attractor

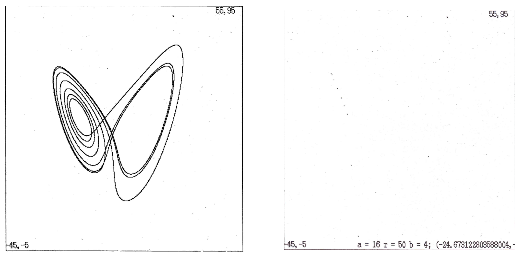




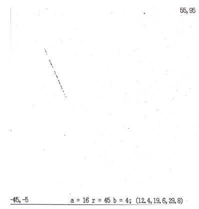
What about a section of a UPO?



?



Aside: finding UPOs



- Section
- Look for close returns
- Cluster
- Average
- See Gunaratne, So papers

Computing sections

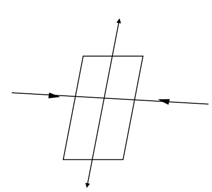
- If you're slicing in state space: use the "inside-outside" function
- If you're slicing in *time*: use modulo on the timestamp

Time-slice sections of periodic orbits: some thought experiments

- pendulum rotating @ 1 Hz and strobe @ 1 Hz?
- pendulum rotating @ 1 Hz and strobe @ 2 Hz?
- pendulum rotating @ 1 Hz and strobe @ 3 Hz?
- pendulum rotating @ 1 Hz and strobe @ 1/2 Hz?
- pendulum rotating @ 1 Hz and strobe @ π Hz? (or some other irrational)

Stability, λ, and the un/stable manifolds

These λ & manifolds play a role in control of chaos…



Lyapunov exponents:

- one λ for each dimension; Σλ < 0 for dissipative systems
- λ are same for all ICs in one basin
- negative λ compress state space along *stable manifolds*
- positive λ stretch it along *unstable manifolds*
- biggest one (λ_1) dominates as t → infinity
- *positive* λ_1 *is a signature of chaos*
- calculating them:
  - From equations: eigenvalues of the variational matrix (see variational system notes on CSCI5446 course webpage; see link from Liz's homepage.)
  - From data: various algorithms that are hideously sensitive to numerics, noise, data length, & algorithmic parameters…

Calculating λ (& other invariants) *from data*

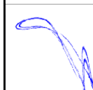
- A good reference: Kantz & Schreiber, *Nonlinear Time Series Analysis* (Abarbanel's book is also very good)
- Associated software: TISEAN www.mpipks-dresden.mpg.de/~tisean

TISEAN
Nonlinear Time Series Analysis

Rainer Hegger
Holger Kantz
Thomas Schreiber

Go to Version 3.0.1 (released March 2007)

Go to Version 2.1 (released December 2000)



TISEAN 3.0.1

TISEAN home

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General Manual

Surrogates Manual

Tutorial

Usage Notes

Installation

Problems

TISEAN 3.0.1: Table of Contents

All programs in alphabetical order

Sections

- Generating time series
- Linear tools
- Utilities
- Stationarity
- Embedding and Poincaré sections
- Prediction
- Noise reduction
- Dimension and entropy estimation
- Lyapunov exponents
- Surrogate data
- Spike trains
- XTisean
- Unsupported

Generating time series

A few routines are provided to generate test data from simple equations. Since there are powerfull packages (for ex
Helena Nusse and Jim Yorke) that can generate chaotic data, we have only included a minimal selection here.

Lyapunov exponents are an important means of quantification for unstable systems. They are however difficult to estimate from a time series. Unless low dimensional, high quality data is at hand, one should not attempt to calculate the full spectrum. Try to compute the maximal exponent first. The two implementations differ slightly. While lyap_k implements the formula by Kantz, lyap_r uses that by Rosenstein et al. which differs only in the definition of the neighbourhoods. We recommend to use the former version, lyap_k.

The estimation of Lyapunov exponents is also discussed in the introduction paper. A recent addition is a programm to compute finite time exponents which are not invariant but contain additional information.

Maximal exponent	lyap_k, lyap_r
Lyapunov spectrum	lyap_spec

Description of the program: lyap_k

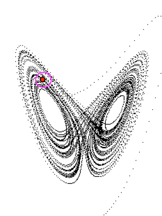
The program estimates the largest Lyapunov exponent of a given scalar data set using the algorithm of Kantz.

Usage:

lyap_k [Options]

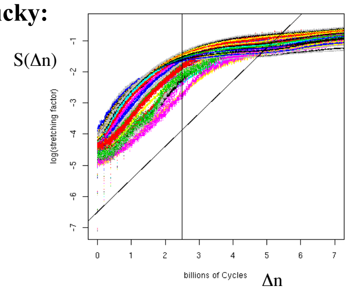
Everything not being a valid option will be interpreted as a potential datafile name. Given no datafile at all, means read stdin. Also - means stdin

Kantz's algorithm:



1. Choose point K •
2. Look at the points around it
3. Measure how far they are from K
4. Average those distances
5. Watch how that average grows with time (Δn)
6. Take the log, normalize over time ➔ S(Δn)
7. Repeat for lots of points K and average the S(Δn)

If you're lucky:



The slope of the scaling region—iff one exists—is the λ_1

Calculating λ (& other invariants) *from data*

- *Be careful! TISEAN has lots of knobs and its results are incredibly sensitive to their values!*
- Use your dynamics knowledge to understand & use those knobs intelligently
- *Look* at the plot; do not blindly fit a regression line to something that has no scaling region

Option	Description	Default
-l#	number of data to be used	whole file
-x#	number of lines to be ignored	0
-c#	column to be read	1
-M#	maximal embedding dimension to use	2
-m#	minimal embedding dimension to use	2
-d#	delay to use	1
-r#	minimal length scale to search neighbors	(data interval)/1000
-R#	maximal length scale to search neighbors	(data interval)/100
-##	number of length scales to use	5
-n#	number of reference points to use	all
-s#	number of iterations in time	50
-t#	'theiler window'	0
-o#	output file name	without file name: 'datafile'.lyap (or stdin.lyap if the data were read from stdin)
-V#	verbosity level 0: only panic messages 1: add input/output messages 2: add statistics for each iteration	3
-h	show these options	none

Description of the Output:

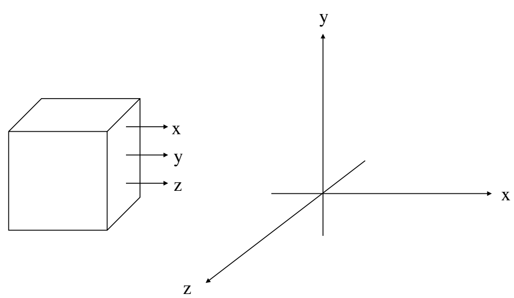
For each embedding dimension and each length scale the file contains a block of data consisting of 3 columns

1. The number of the iteration
2. The logarithm of the stretching factor (the slope is the Lyapunov exponent if it is a straight line)
3. The number of points for which a neighborhood with enough points was found

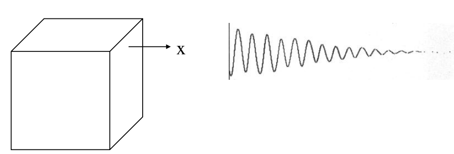
Fractal dimension:

- Capacity
- Box counting
- Correlation (d2 in TISEAN)
- Lots of others:
  - Kth nearest neighbor
  - Similarity
  - Information
  - Lyapunov
  - …
- See Chapter 6 and §11.3 of Kantz & Schreiber

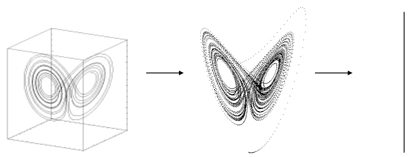
We've been assuming that we can measure all the state variables…



But often you can't.



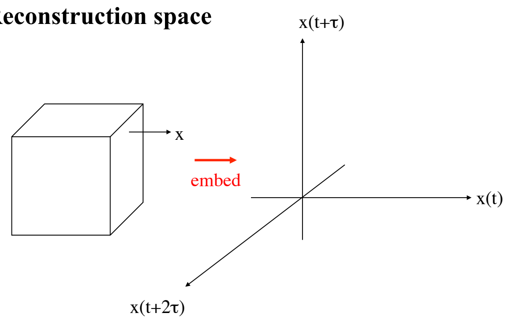
How to undo a projection?



Delay-coordinate embedding

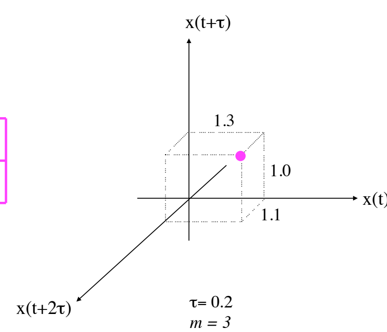
"reinflate" that squashed data to get a *topologically identical* copy of the original thing.

Reconstruction space



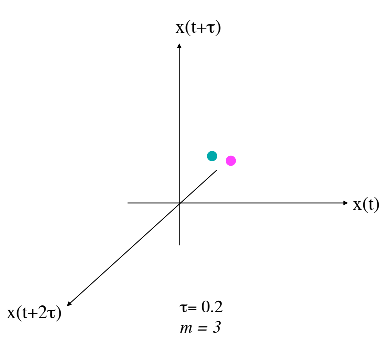
Mechanics

x	t
1.3	0.1
1.2	0.2
1.0	0.3
0.8	0.4
1.1	0.5
1.4	0.6
1.6	0.7



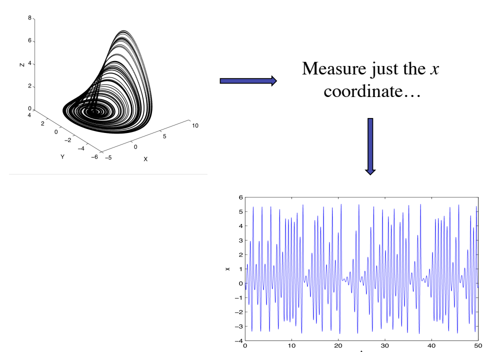
τ= 0.2
m = 3

x	t
1.3	0.1
1.2	0.2
1.0	0.3
0.8	0.4
1.1	0.5
1.4	0.6
1.6	0.7

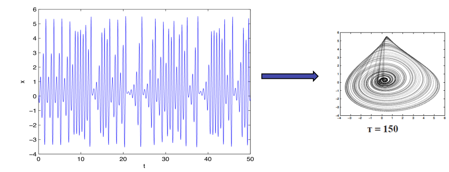


τ= 0.2
m = 3

TISEAN's `delay` command does this

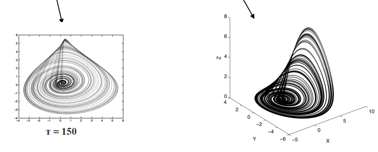


…and then embed:



Takens* theorem

For the right τ and enough dimensions, the embedded dynamics are diffeomorphic to (have same topology as) the original state-space dynamics.



* Whitney, Mane, …

Note: the measured quantity must be a smooth, generic function of at least one state variable, and must be uniformly sampled in time.

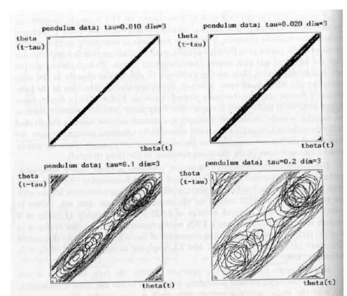
Diffeomorphisms and topology

Diffeomorphic: mapping from the one to the other is differentiable and has a differentiable inverse.

What that means:

- *qualitatively* the same shape
- have same dynamical invariants (e.g., λ)

Picking τ



TISEAN contains tools that help you do this (e.g., `mutual`)

Picking *m*

$m > 2d$: sufficient to ensure no crossings in reconstruction space:

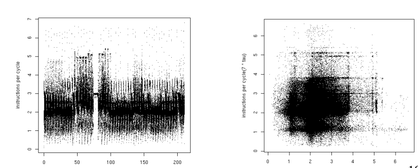
…may be overkill.

"Embedology" paper: $m > 2\ d_{box}$ (box-counting dimension)

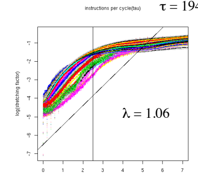
TISEAN contains tools that help you do this (e.g., `false_nearest`)

NLTSA* of computer performance dynamics

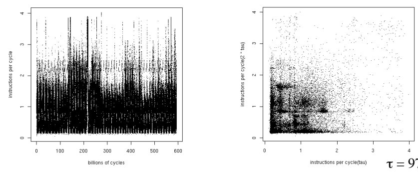
* nonlinear time-series analysis

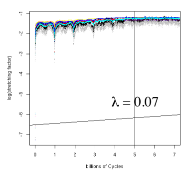


`bzip2` dynamics on an Intel Core2



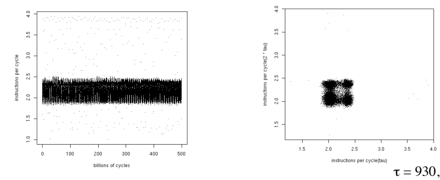
Mytkowicz *et al.*, *Chaos* **19**:033124

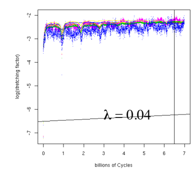




`bzip2` dynamics on an Intel Pentium 4

Mytkowicz *et al.*, *Chaos* **19**:033124





`povray` dynamics on an Intel Core2

Mytkowicz *et al.*, *Chaos* **19**:033124

Caveat: need enough data…

If Δt is not uniform

Theorem (Takens): for $\tau>0$ and $m > 2d$, reconstructed trajectory is diffeomorphic to the true trajectory

Conditions: evenly sampled in time

Interspike interval embedding

<u>idea</u>: lots of systems generate spikes — hearts, nerves, etc.

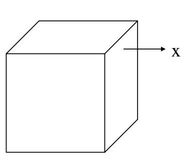
if you assume that the spikes are the result of an integrate-and-fire system, then the Δt has a one-to-one correspondence to some state variable's *integrated* value…

<u>*in which case the Takens theorem still holds.*</u>

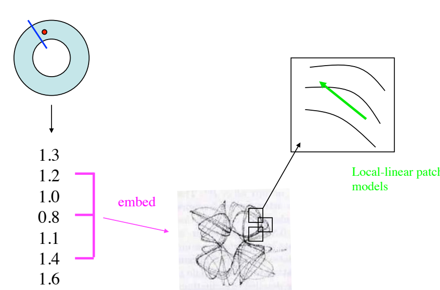
(with the Δts as state variables)

Sauer *Chaos* **5**:127

What if we measured time-series data from a roulette wheel?

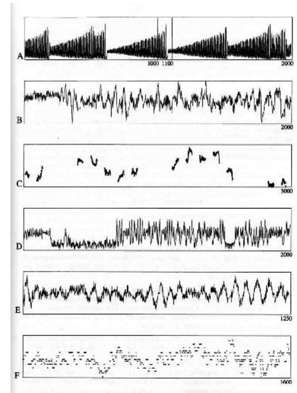


<u>The Eudaemonic Pie</u>
(or <u>The Newtonian Casino</u>)



The Santa Fe competition

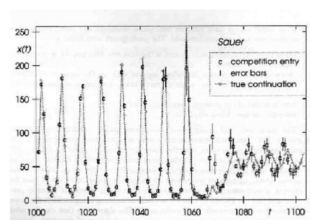
- Weigend & Gershenfeld, 1992
- put a bunch of data sets up on an ftp server
- and invited all comers to predict their future
- chronicled in *Time Series Prediction: Forecasting the Future and Understanding the Past,* Santa Fe Institute, 1993 (from which the images on the following half-dozen slides were reproduced)



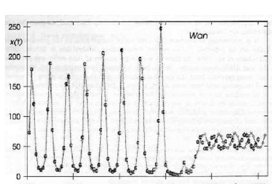
The Santa Fe competition: data

- Laboratory laser
- Medical data (sleep apnea)
- Currency rate exchange
- RK4 on some chaotic ODE
- Intensity of some star
- A Bach fugue

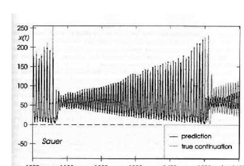
Embedding + patch models: (Sauer)



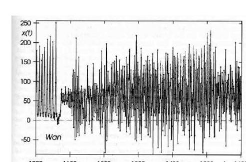
Neural net: (Wan)



Further out:

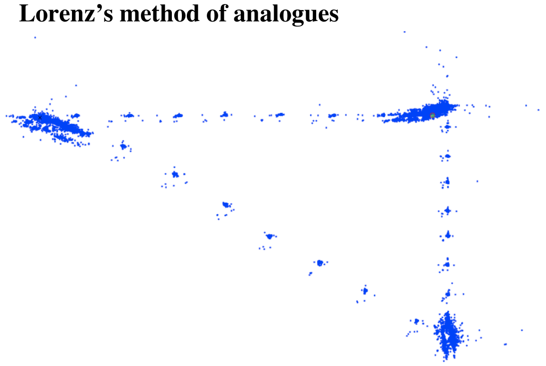


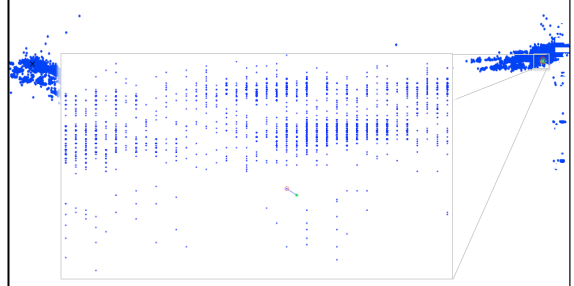
Sauer



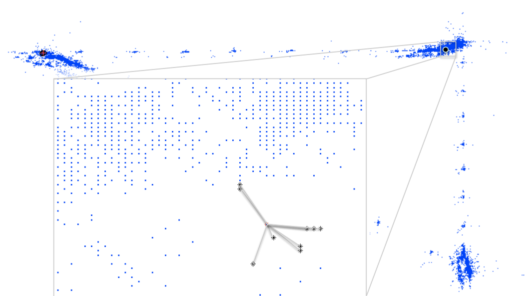
Wan

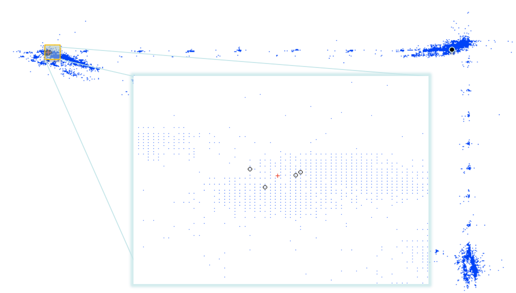
Lorenz's method of analogues

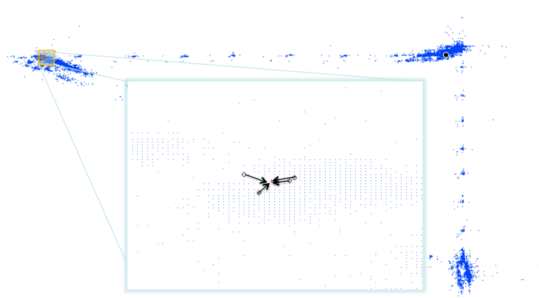




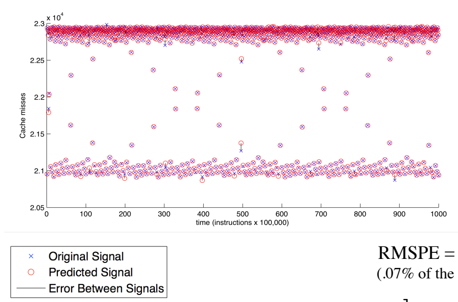
A *k*-nearest neighbor modification of LMA







Using *k*LMA to predict computer dynamics



RMSPE = 11.566
(.07% of the average)

`column_major`
cache misses

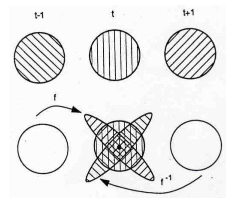
Garland/Bradley *Intelligent Data Analysis* 2011

Noise…

Linear filtering: a bad idea if the system is chaotic

Nonlinear alternatives:

- use the stable and unstable manifold structure on a chaotic attractor…

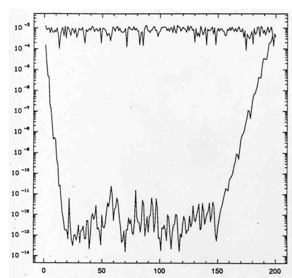


Farmer & Sidorowich, in *Evolution, Learning and Cognition*, World Scientific, 1983

Idea:

- If you have a model of the system, you can simulate what happens to each point in forward *and backward* time
- If your system has transverse stable and unstable manifolds, that does useful things to the noise balls
- Since all three versions of that data should be identical at the middle time, can average them
- ⇨ noise reduction!
- Works best if manifolds are perpendicular, but requires only transversality

Results:



Farmer & Sidorowich, in *Evolution, Learning and Cognition*, World Scientific, 1983