So far: mostly about maps.

- discrete time systems:
  - time proceeds in clicks
  - “maps”
  - modeling tool: difference equation

Next up: flows

- continuous time systems:
  - time proceeds smoothly
  - “flows”
  - modeling tool: differential equations

Attractors

- Attractors exist only in dissipative systems!
- Dissipation $\Leftrightarrow$ contraction of state space under the influence of the dynamics
- Can still have chaos if no dissipation…just not chaotic attractors

Conditions for chaos in continuous-time systems

**Necessary:**
- Nonlinear
- At least three state-space dimensions \( \text{(NB: only one needed in maps)} \)

**Sufficient:**
- “Nonintegrable”
  - i.e., cannot be solved in closed form

Concepts: review

- State variable
- State space
- Initial condition
- Trajectory
- Attractor
- Basin of attraction
- Transient
- Fixed point (un/stable)
- Bifurcation
- Parameter
A cool Lorenz applet:

http://www.exploratorium.edu/complexity/java/lorenz.html

(Note: by Jim Crutchfield, another SFI person, who will be here at the end of next week)

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Deterministic Nonperiodic Flow

Edward N. Lorenz

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Manuscript received 15 November 1963, in revised form 1 January 1964

Abstract

Three equations of deterministic ordinary nonlinear differential equations may be designed to represent forced, dissipative, hydrodynamic flow. Solutions of these equations can be initialized with latitudes to please users. For these systems with bounded solutions, such that nonperiodic solutions are readily obtained numerically, small changes in initial conditions lead to large differences in solutions. Systems with bounded solutions are shown to possess bounded nonperiodic solutions. A single system representing mid-latitude convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The possibility of very long-range weather prediction is examined in the light of these results.

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• Equations:

  \[ x' = a(y-x) \]
  \[ y' = rx - y - xz \]
  \[ z' = xy - bz \]

  (first three terms of a Fourier expansion of the Navier-Stokes eqns)

• State variables:
  • \( x \) convective intensity
  • \( y \) temperature
  • \( z \) deviation from linearity in the vertical convection profile
• Parameters:
  • $a$ Prandtl number - fluids property
  • $r$ Rayleigh number - related to $\Delta T$
  • $b$ aspect ratio of the fluid sheet

$x' = 16(y-x)$
$y' = 45x - y - xz$
$z' = xy - 4z$
Maybe add Donny's Lorenz movie here?

See student work folder in directory above

Courtesy of Donny Warbritton

Attractors

Four types:

• fixed points
• limit cycles (aka periodic orbits)
• quasiperiodic orbits
• chaotic attractors

A nonlinear system can have any number of attractors, of all types, sprinkled around its state space.

Their basins of attraction (plus the basin boundaries) partition the state space.

And there’s no way, a priori, to know where they are, how many there are, what types, etc.

Attractors

• Fixed point

Attractors

• Limit cycle
Attractors

- Quasi-periodic orbit...

"Strange" or chaotic attractors

- Often fractal
- Covered densely by trajectories
- Exponential divergence of neighboring trajectories...

Lyapunov exponents

- Nonlinear analogs of eigenvalues: one $\lambda$ for each dimension

Lyapunov exponents: summary

- Nonlinear analogs of eigenvalues: one $\lambda$ for each dimension
- Negative $\lambda$ compress state space; positive $\lambda$, stretch it
- $\Sigma \lambda < 0$ for dissipative systems
- Long-term average in definition; biggest one dominates as $t \to \infty$
- Positive $\lambda$ is a signature of chaos
- $\lambda_i$ are same for all ICs in one basin

"Strange" or chaotic attractors:

- Exponential divergence of neighboring trajectories
- Often fractal
- Covered densely by trajectories
- Contain an infinite number of "unstable periodic orbits"…

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Unstable periodic orbits (UPOs)

Attractor “bones”…

Poincare recurrence