Patterns of Synchrony
From Animal Gaits to Binocular Rivalry

Gateways to Emergent Behavior in Science and Society
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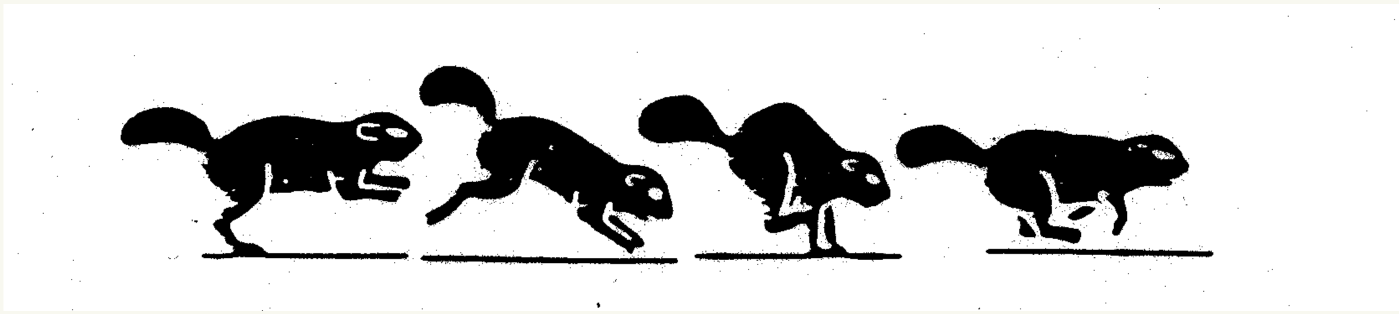
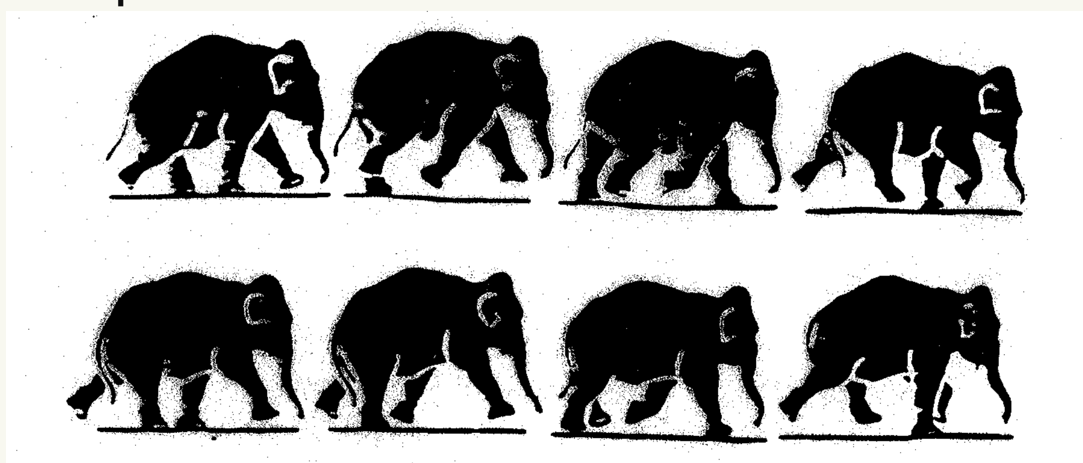
Quadruped Gaits

- Bound of the Siberian Souslik

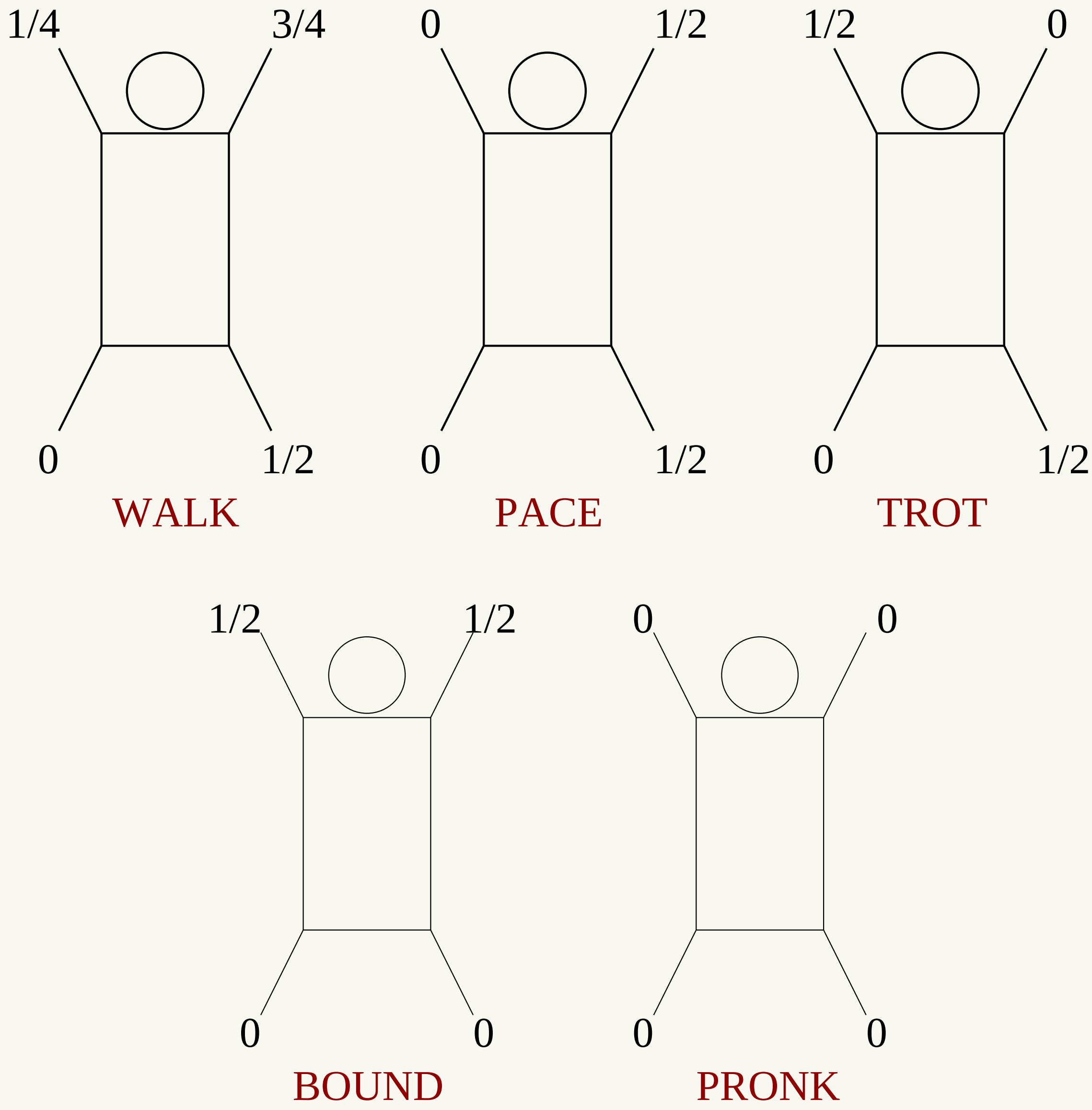


- Amble of the Elephant



- Trot of the Horse

Rigid Gait Phases for Quadruped Gaits



Gait Symmetries / Central Pattern Generators

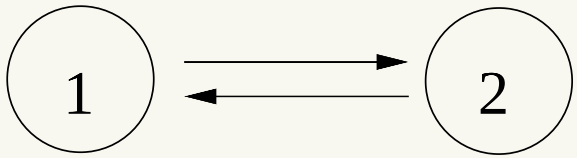
Gait	Spatio-temporal symmetries		
Trot	(Left/Right, $\frac{1}{2}$)	and	(Front/Back, $\frac{1}{2}$)
Pace	(Left/Right, $\frac{1}{2}$)	and	(Front/Back, 0)
Walk	(Figure Eight, $\frac{1}{4}$)		

- Network of neurons (CPG) that produces gait rhythms
- Hodgkin - Huxley (1952)
  Neuron modeled by system of differential equations
- Design simplest network that produces independent

walk, trot, pace

Collins and Stewart (1993); G., Stewart, Buono, and Collins (1999)

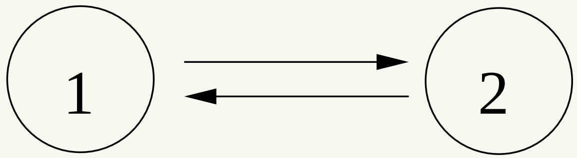
Rigid Phase-Shift Synchrony



$$\begin{aligned}\dot{x}_1 &= f(x_1, x_2)\\ \dot{x}_2 &= f(x_2, x_1)\end{aligned} \qquad x_1, x_2 \in \mathbf{R}^k$$

- Symmetry: sends solutions to solutions $\mathbf{Z}_2$ symmetry
- Nodes oscillate in phase: $x_2(t) = x_1(t)$
  Nodes half-period out of phase: $x_2(t) = x_1(t + \frac{1}{2}T)$

Rigid Phase-Shift Synchrony



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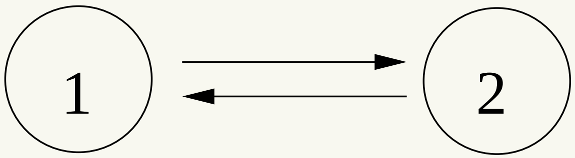
- Let $x(t)$ be a hyperbolic T-periodic solution

  $H = \{\gamma \in \Gamma : \gamma\{x(t)\} = \{x(t)\}\}$ **spatiotemporal symmetries**

  $\gamma \in H \Longrightarrow \theta \in [0, 1)$ such that $\gamma x(t) = x(t + \theta T)$

Rigid Phase-Shift Synchrony



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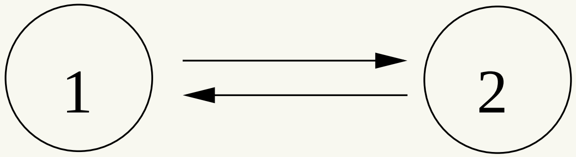
  $\gamma \in H \Longrightarrow \theta \in [0, 1)$ such that $\gamma x(t) = x(t + \theta T)$
- H is rigid to equivariant perturbations
- Example: $H = \mathbf{Z}_2(1\ 2); \quad \theta = 0$ or $\theta = \frac{1}{2}$

Two Identical Cells: Solutions via Hopf Bifurcation



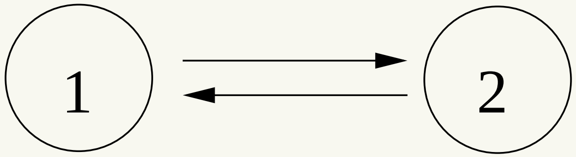
$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, \lambda) \\ \dot{x}_2 &= f(x_2, x_1, \lambda) \\ 0 &= f(0, 0, \lambda) \end{aligned} \qquad x_1, x_2 \in \mathbf{R}^k$$

- $J(\lambda) = \begin{bmatrix} \alpha(\lambda) & \beta(\lambda) \\ \beta(\lambda) & \alpha(\lambda) \end{bmatrix}$; $\begin{bmatrix} x \\ x \end{bmatrix}$, $\begin{bmatrix} x \\ -x \end{bmatrix}$ invariant subsp's

  $\alpha =$ linear internal dynamics; $\beta =$ linear coupling

- eigenvalues of J are eigenvalues of $\alpha + \beta$ and $\alpha - \beta$

Two Identical Cells: Solutions via Hopf Bifurcation



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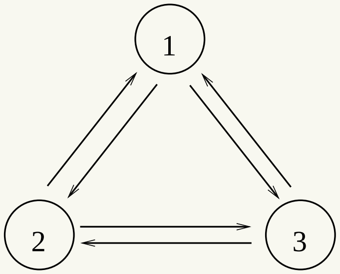
  $\alpha =$ linear internal dynamics; $\beta =$ linear coupling

- eigenvalues of J are eigenvalues of $\alpha + \beta$ and $\alpha - \beta$

- $\alpha + \beta$ critical: synchronous periodic solutions

- $\alpha - \beta$ critical: half-period out of phase periodic solutions

Three-Cell Bidirectional Ring: $\Gamma = \mathbf{D}_3$



$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_3) \\ \dot{x}_2 &= f(x_2, x_3, x_1) \\ \dot{x}_3 &= f(x_3, x_1, x_2) \end{aligned} \qquad f(x_2, x_1, x_3) = f(x_2, x_3, x_1)$$

- Discrete rotating waves: $H = \mathbf{Z}_3(1\ 2\ 3)$, $\theta = \frac{1}{3}$

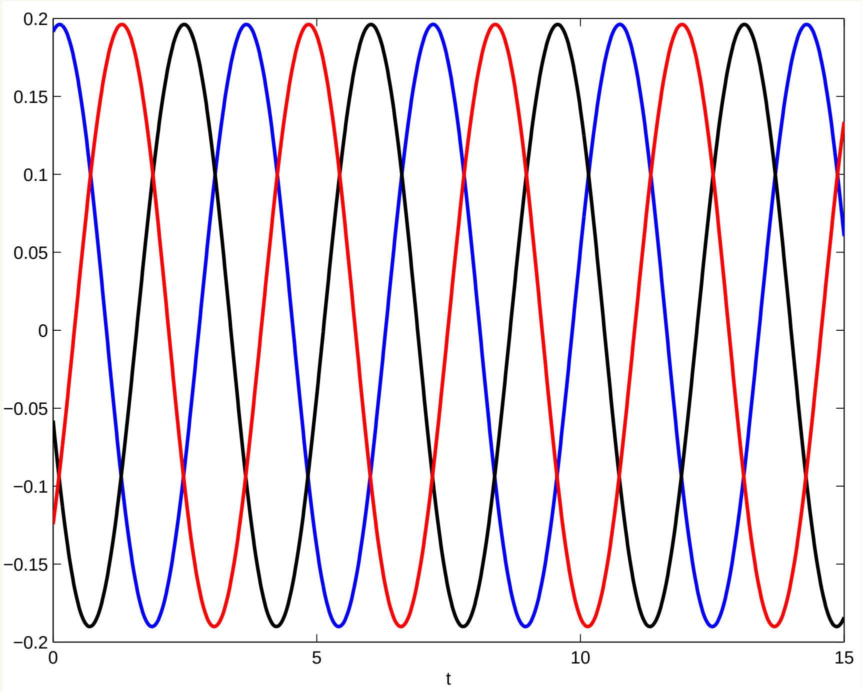
$$x_3(t) = x_2(t + \tfrac{1}{3}T) = x_1(t + \tfrac{2}{3}T)$$

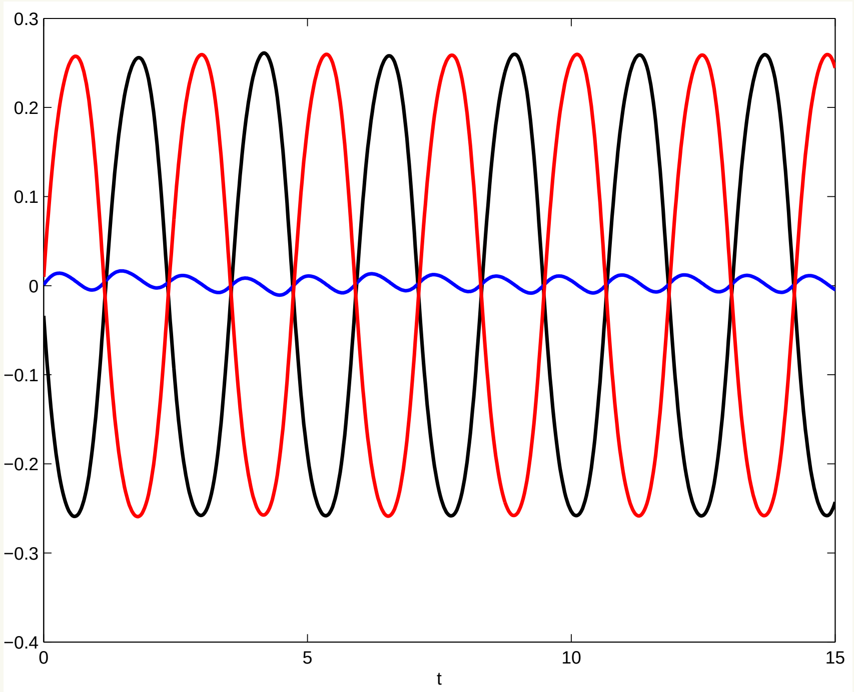
- Out-of-phase periodic solutions: $H = \mathbf{Z}_2(1\ 3)$, $\theta = \frac{1}{2}$

$$x_3(t) = x_1\left(t + \tfrac{1}{2}T\right) \quad \text{and} \quad x_2(t) = x_2\left(t + \tfrac{1}{2}T\right)$$

G. and Stewart (1986); van Gils and Valkering (1986)

Time Series and Phase Shifts





- Discrete Rotating Wave: $x_3(t) = x_2\left(t + \frac{1}{3}T\right) = x_1\left(t + \frac{2}{3}T\right)$
- Out-of-phase: $x_3(t) = x_1\left(t + \frac{1}{2}T\right)$ and $x_2(t) = x_2\left(t + \frac{1}{2}T\right)$

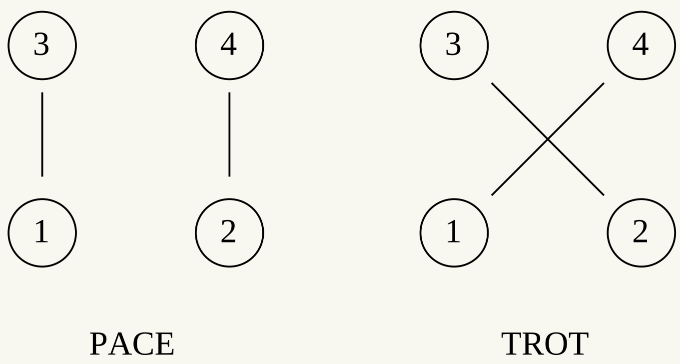
G. and Stewart (1986); van Gils and Valkering (1986)

Four Cells Do Not Suffice

- Network produces walk. There is a four-cycle symmetry

$$(1\ 3\ 2\ 4)$$

- Four-cycle permutes pace to trot



- CPG cannot be modeled by four-cell network where each cell gives rhythmic pulsing to one leg

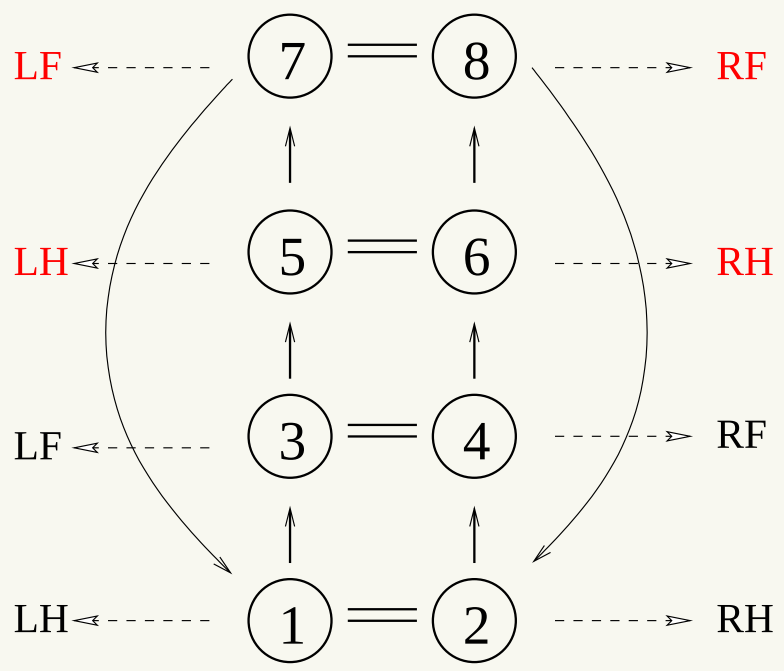
G., Stewart, Buono, and Collins (1999)

Central Pattern Generators (CPG)

- Use gait symmetries to construct network and rhythms

  1) walk $\Longrightarrow$ four-cycle ω in symmetry group
  2) pace or trot $\Longrightarrow$ transposition κ in symmetry group

- Simplest network has $\mathbf{Z}_4(\omega) \times \mathbf{Z}_2(\kappa)$ symmetry



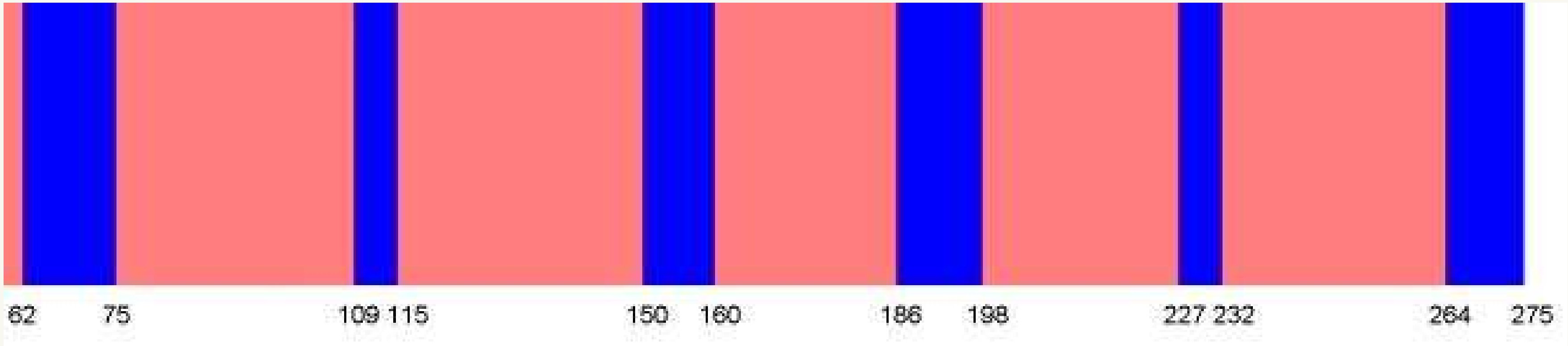
G., Stewart, Buono, and Collins (1999); Buono and G. (2001)

Primary Gaits or Hopf Bifurcation from Stand: $H = \mathbf{Z}_4(\omega) \times \mathbf{Z}_2(\kappa)$

Kernel of H	Phase Diagram	Gait
$\mathbf{Z}_4(\omega) \times \mathbf{Z}_2(\kappa)$	$\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$	pronk
$\mathbf{Z}_4(\omega)$	$\begin{matrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{matrix}$	pace
$\mathbf{Z}_4(\kappa\omega)$	$\begin{matrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{matrix}$	trot
$\mathbf{Z}_2(\kappa) \times \mathbf{Z}_2(\omega^2)$	$\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{matrix}$	bound
$\mathbf{Z}_2(\kappa\omega^2)$	$\begin{matrix} \frac{1}{4} & \frac{3}{4} \\ 0 & \frac{1}{2} \end{matrix}$	walk
$\mathbf{Z}_2(\kappa)$	$\begin{matrix} 0 & 0 \\ \frac{1}{4} & \frac{1}{4} \end{matrix}$	jump

G., Stewart, Buono, and Collins (2000)

The Jump

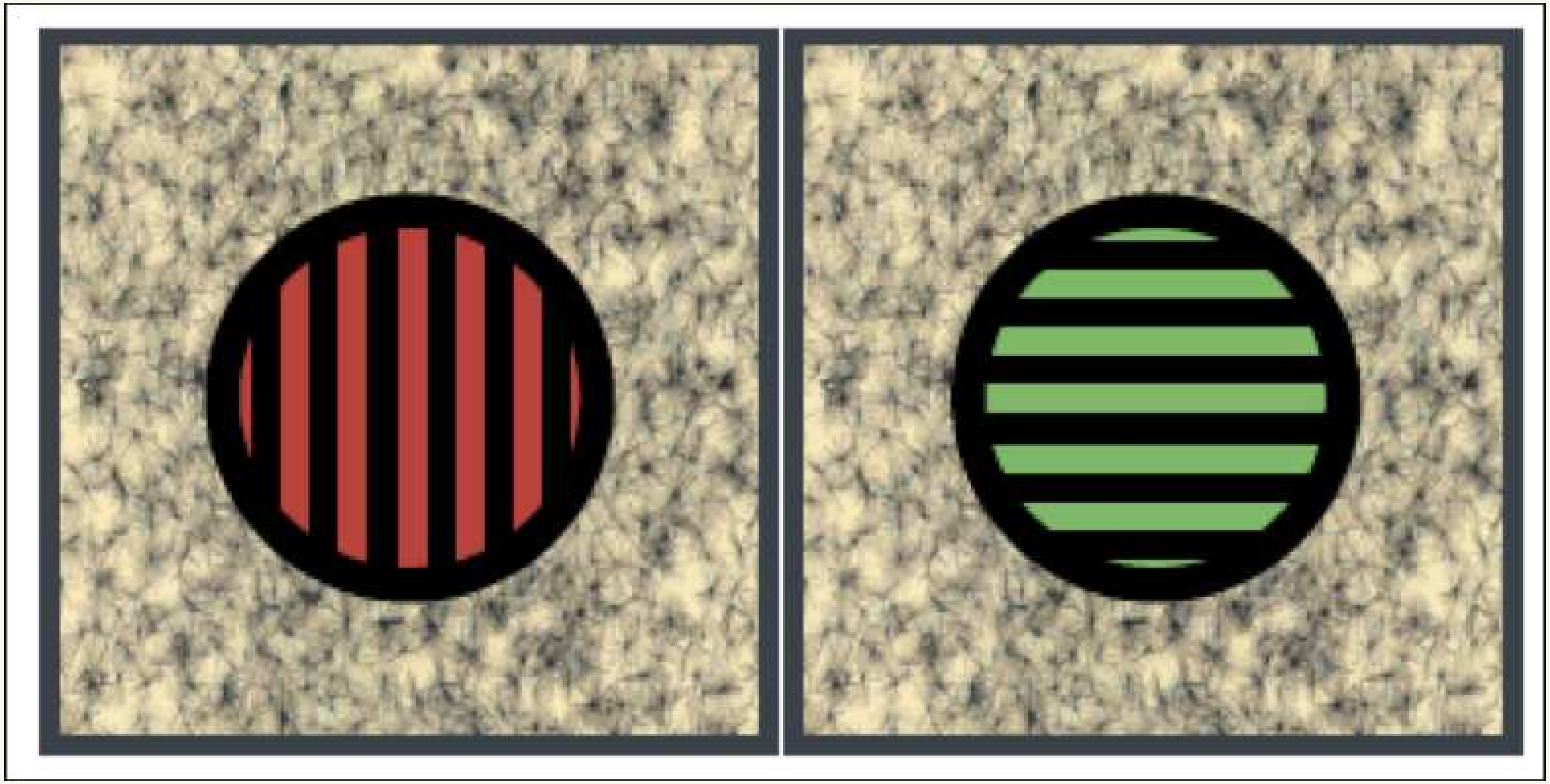


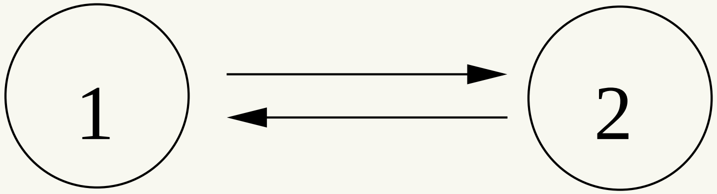
- Average Right Rear to Right Front = 31.2 frames
- Average Right Front to Right Rear = 11.4 frames
- $\frac{31.2}{11.4} = 2.74$

G., Stewart, Buono, and Collins (2000)

Binocular Rivalry: Different Images Presented to Two Eyes

How does the brain deal with CONTRADICTORY information





- Often modeled by two units

Simplest Rivalry Equations for Two Units

- Units a and b consist of an activity variable $*^E$ (firing rate) and a fatigue variable $*^H$ (reduces activity on long time scale)

$$
\begin{aligned}
\varepsilon\, \dot{a}^E &= -a^E + \mathcal{G}\left(I - \beta b^E - g a^H\right) \\
\dot{a}^H &= a^E - a^H \\
\varepsilon\, \dot{b}^E &= -b^E + \mathcal{G}\left(I - \beta a^E - g b^H\right) \\
\dot{b}^H &= b^E - b^H
\end{aligned}
$$

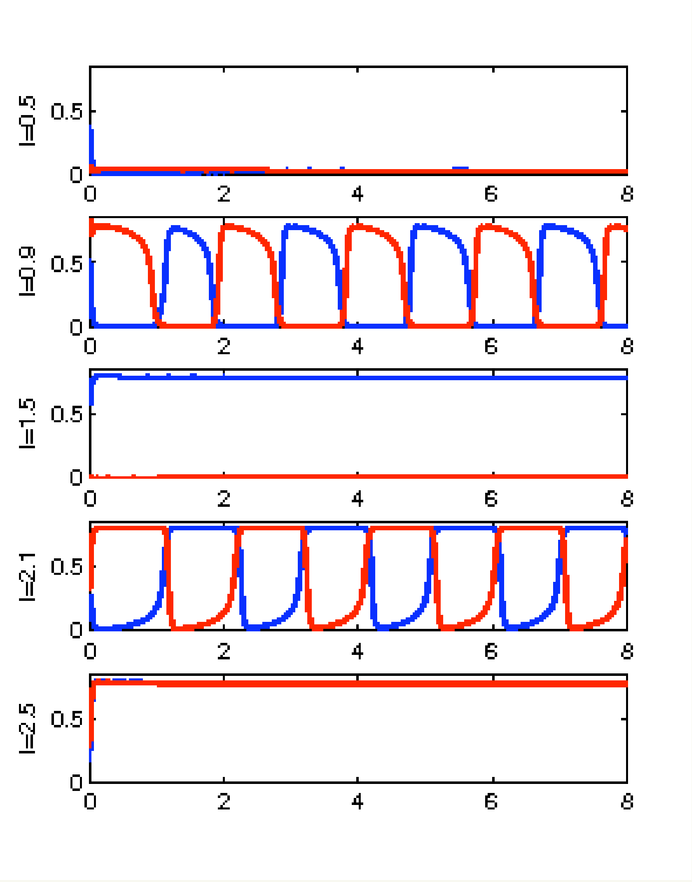
- $\mathcal{G} =$ gain function
  $\beta =$ reciprocal inhibition
  $I =$ external signal strength
  $g =$ strength of reduction of $*^E$ by $*^H$
  $\varepsilon \ll 1$ is ratio of time scales on which $*^E$ and $*^H$ evolve

Laing and Chow (2002), Curtu, Shpiro, Rubin, and Rinzel (2008); Wilson (2009); Curtu (2010); Diekman, G., McMillen, and Wang (2012)

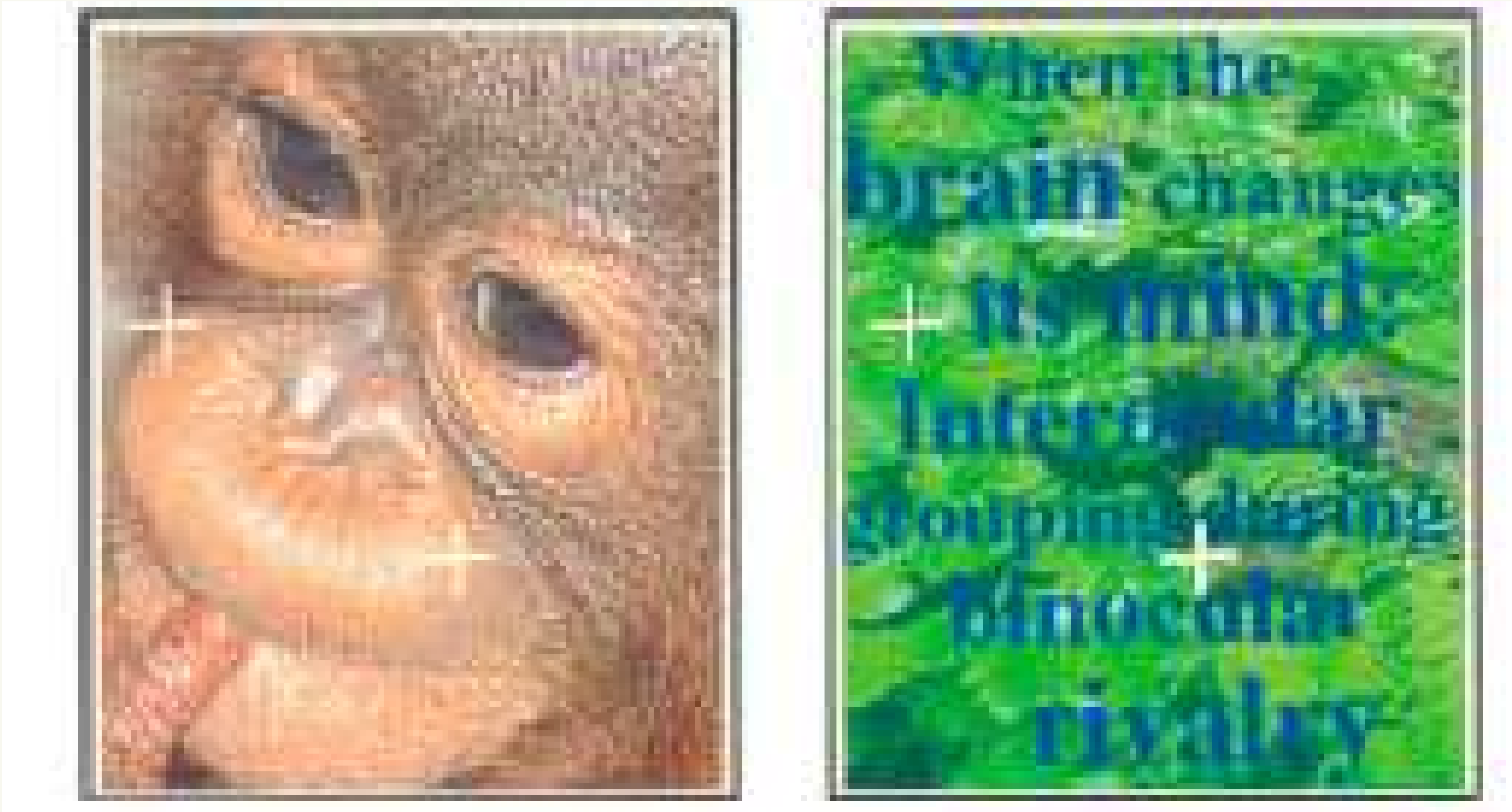
Two Unit Model Solution Types

Three types of states:

- Fusion = equilibria in which units have equal values
  Rigid fused states forced by symmetry
- Winner-Take-All = equilibria with different activity levels
- Rivalry = two units oscillate in periods of dominance
  Rigid rivalry forced by symmetry



Kovács First Experiment: Conventional Monkey and Text

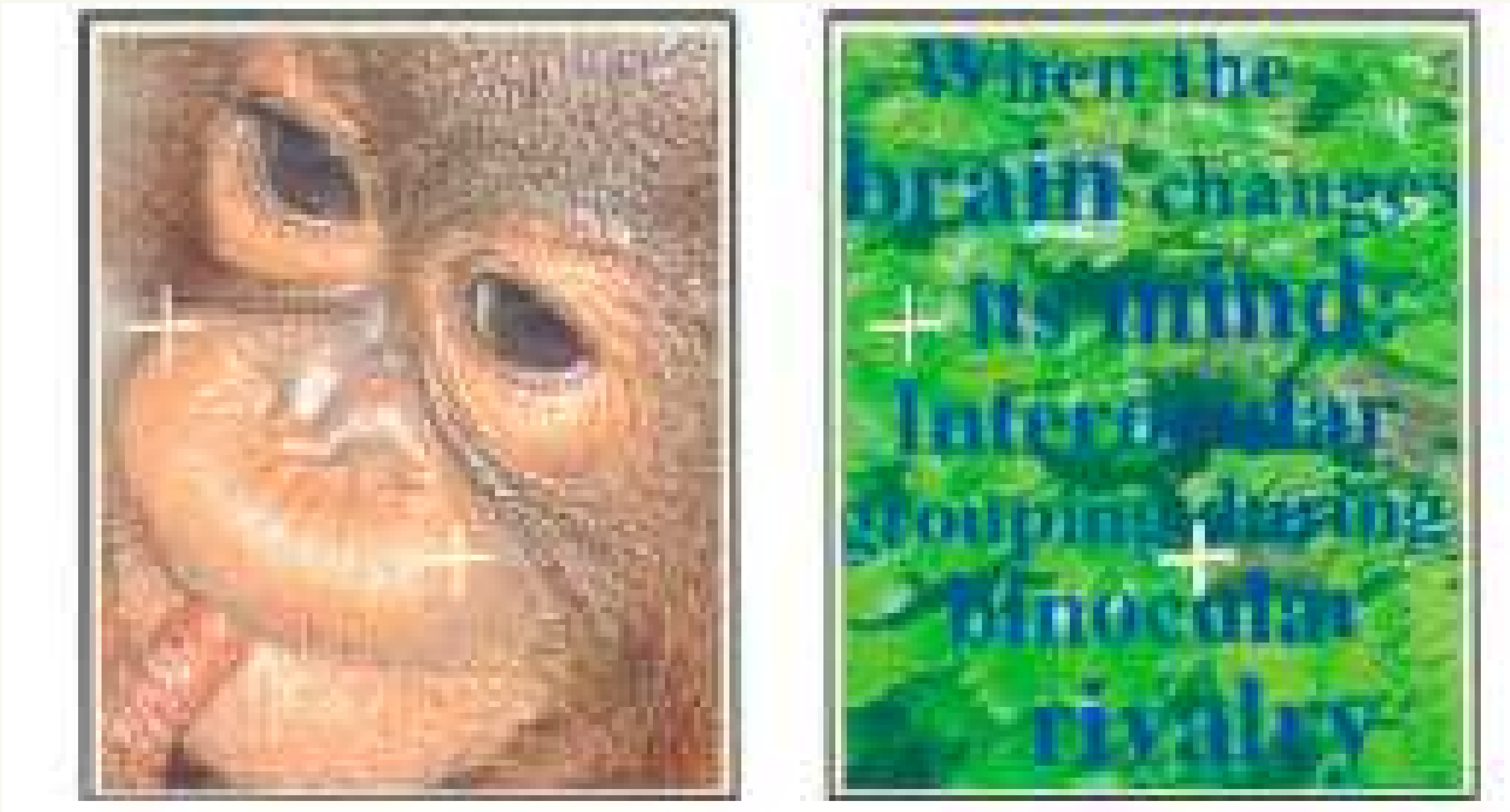


Kovács, Papathomas, Yang, and Fehér (1996)

Kovács Second Experiment: Scrambled Monkey and Text

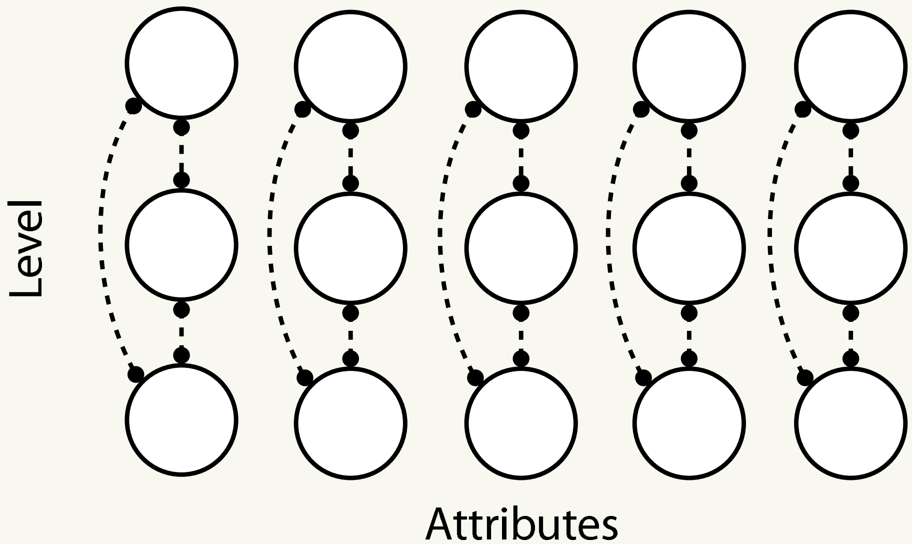
Kovács, Papathomas, Yang, and Fehér (1996)

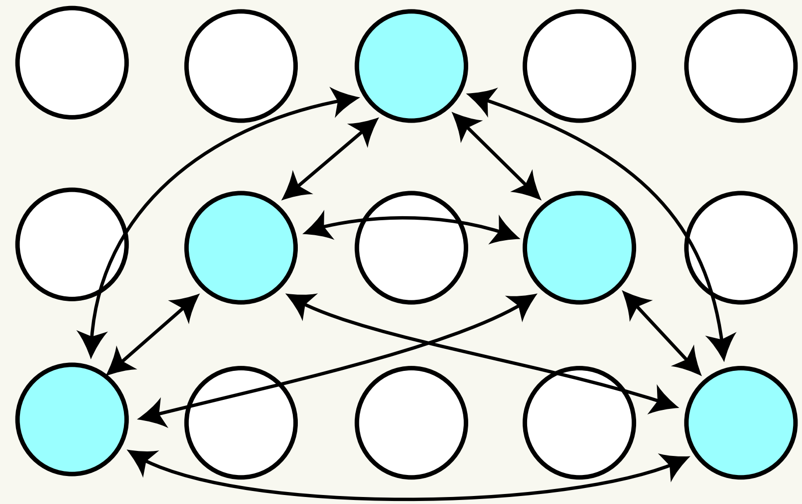
Scrambled Monkey and Text: Interocular Groupings



Kovács, Papathomas, Yang, and Fehér (1996)

Wilson's Model for Generalized Rivalry

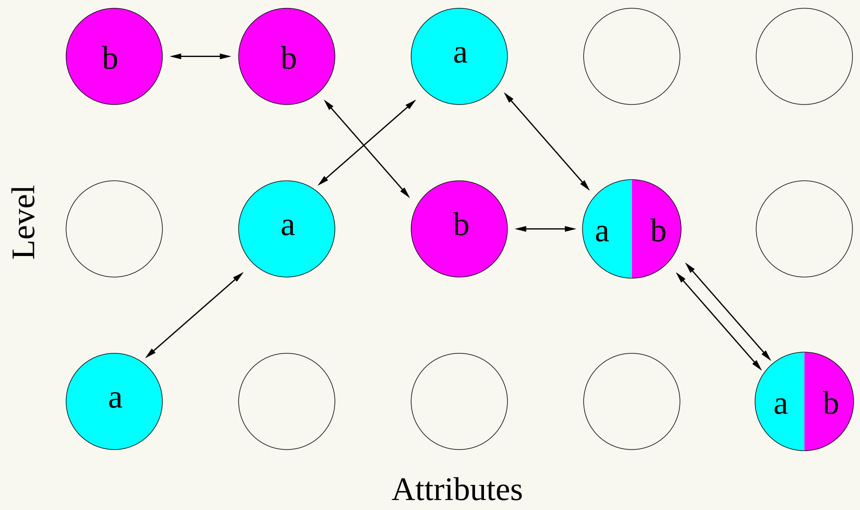




- Columns represent **attributes**; rows represent **level** of attribute
- (L) Inhibition between nodes in column (dashed lines)
- (R) Excitation between nodes in **learned** pattern (solid lines)

Wilson (2008, 2009)

Two Learned Patterns a and b

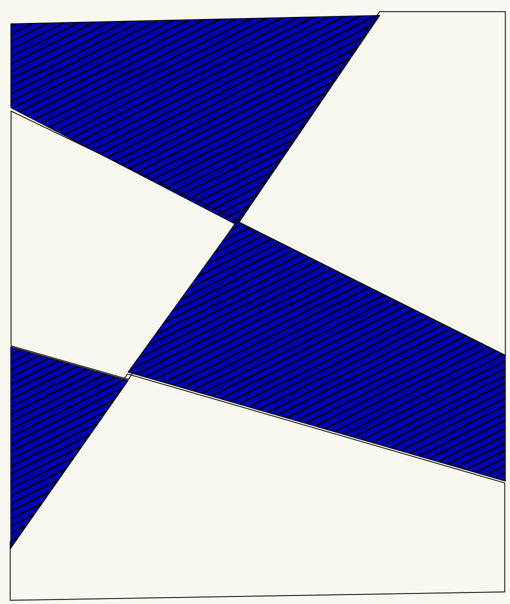


- **Learned pattern**: one node from each attribute column
  Excitation between nodes in learned pattern

- **Derived Patterns**: Patterns that are not learned

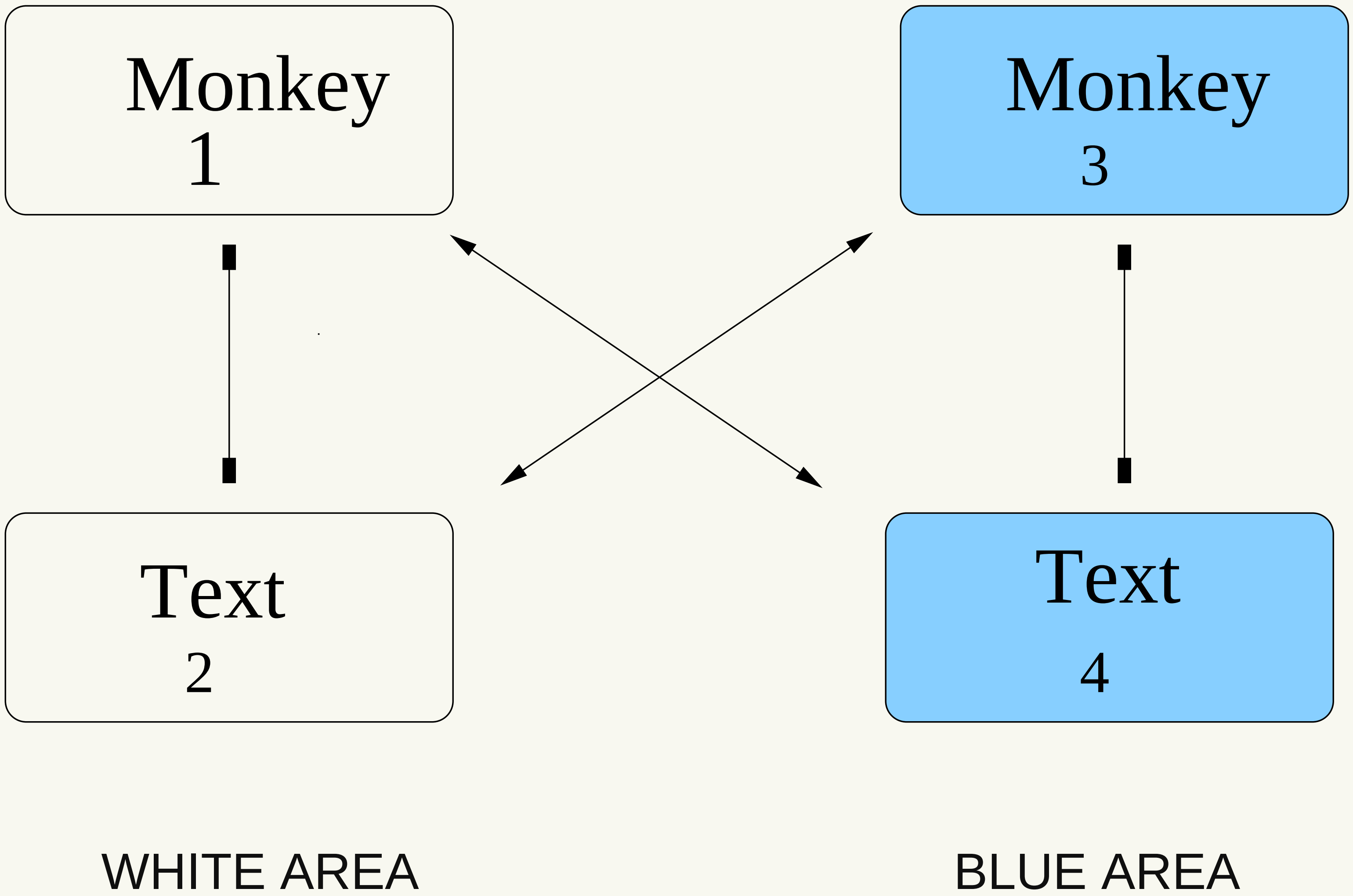
Diekman, G., McMillen, and Wang (2012, 2013)

Rivalry in Monkey and Text

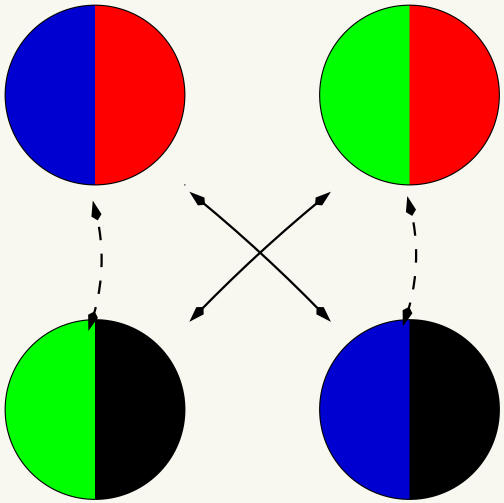


Kovács, Papathomas, Yang, and Fehér (1996)

Wilson Network for Second Kovacs Experiment

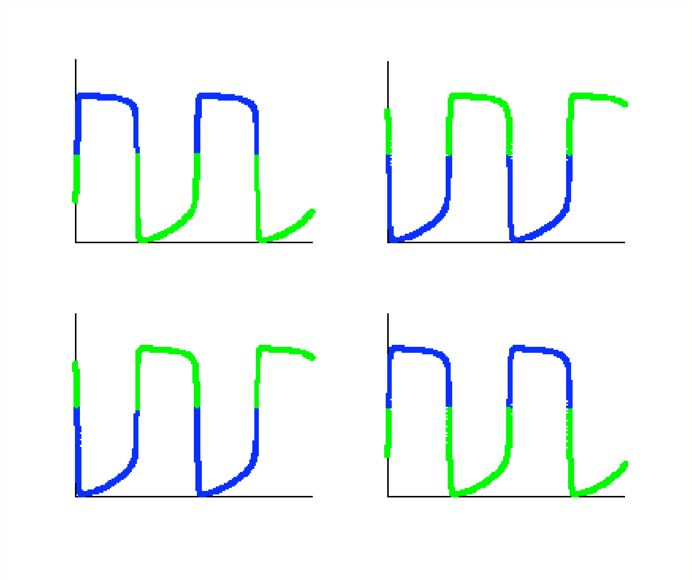


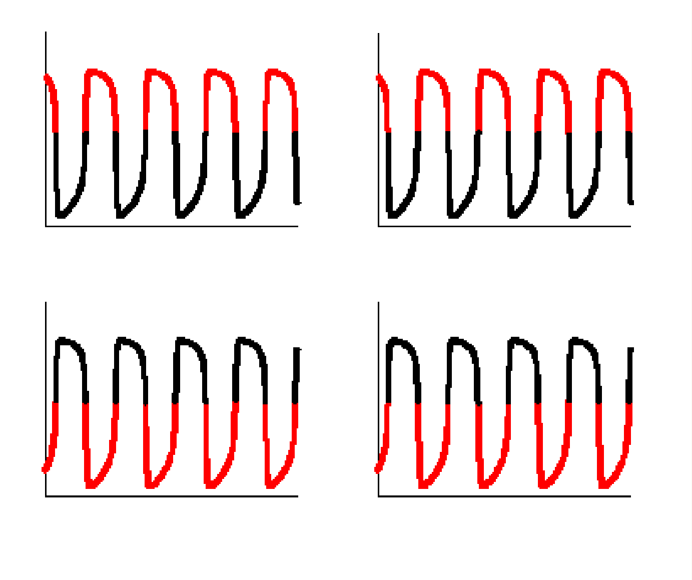
2 Attributes; 2 Levels; 2 Learned Patterns; 2 Derived Patterns



- $x_1^E > x_2^E$ and $x_3^E > x_4^E$ (whole monkey; **derived**, RED)
- $x_1^E > x_2^E$ and $x_4^E > x_3^E$ (mixed image; **learned**, BLUE)
- $x_2^E > x_1^E$ and $x_3^E > x_4^E$ (mixed image; **learned**, GREEN)
- $x_2^E > x_1^E$ and $x_4^E > x_3^E$ (all text; **derived**, BLACK)

Patterns in Wilson Model of Kovacs Second Experiment



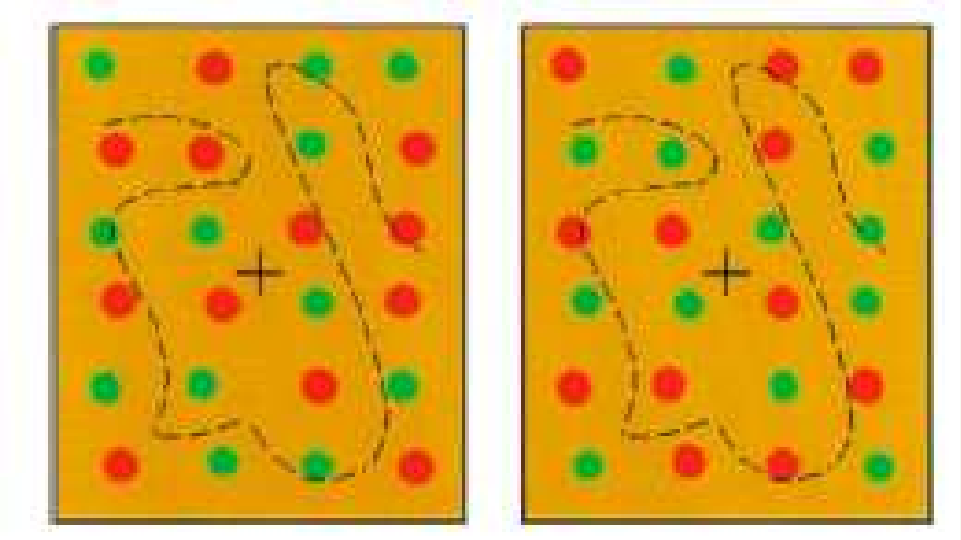


- 2 **learned** patterns: BLUE and GREEN

  2 **derived** patterns: RED and BLACK

- Rivalry: BLUE-GREEN (**learned**); BLACK-RED (**derived**)

Third Kovacs Experiment: Scrambled Disc Patterns



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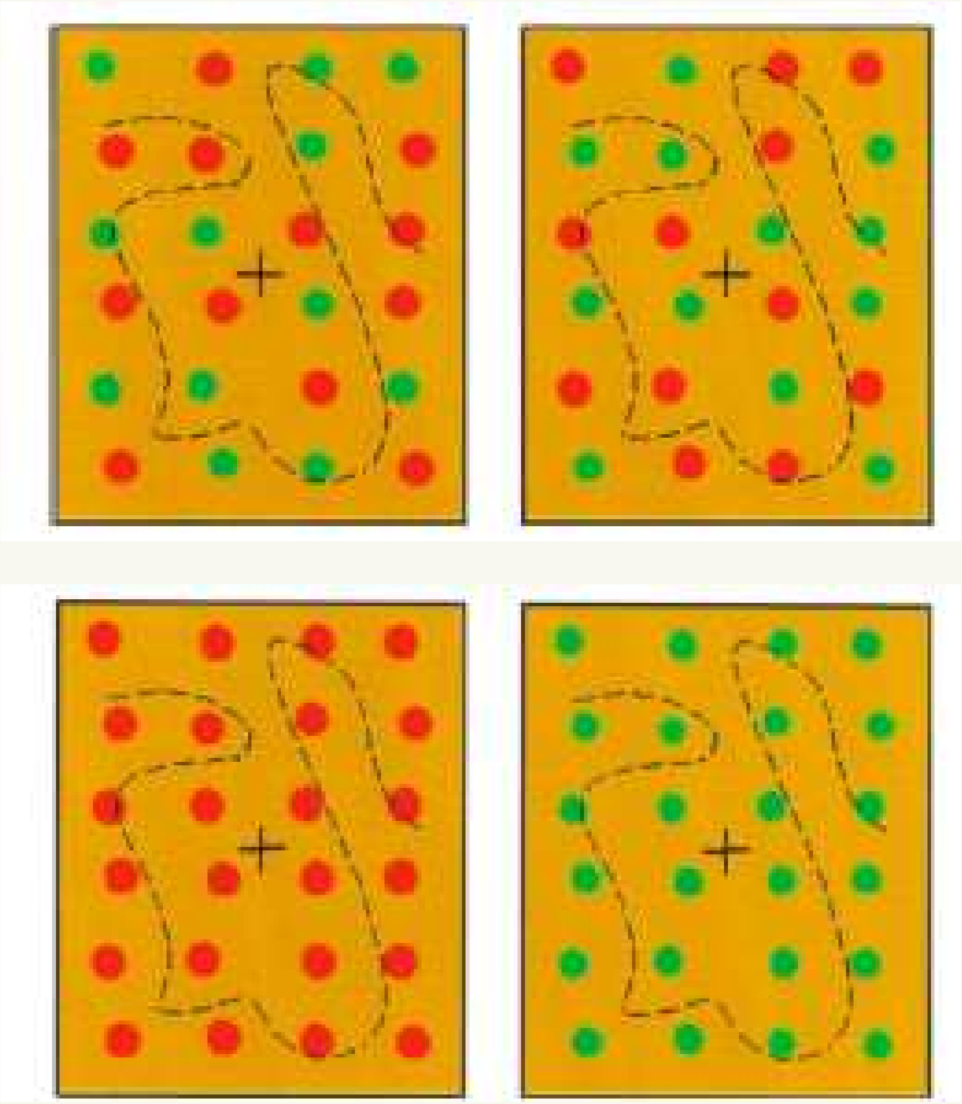
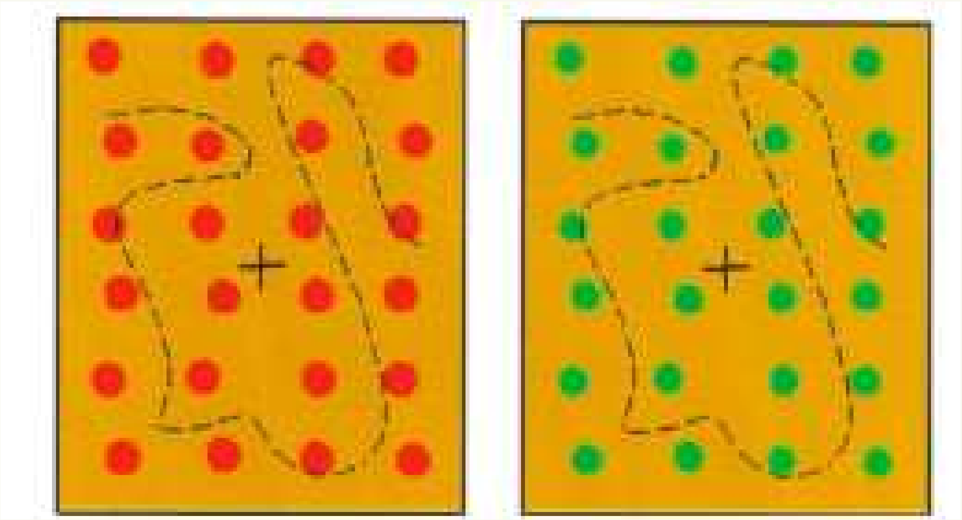
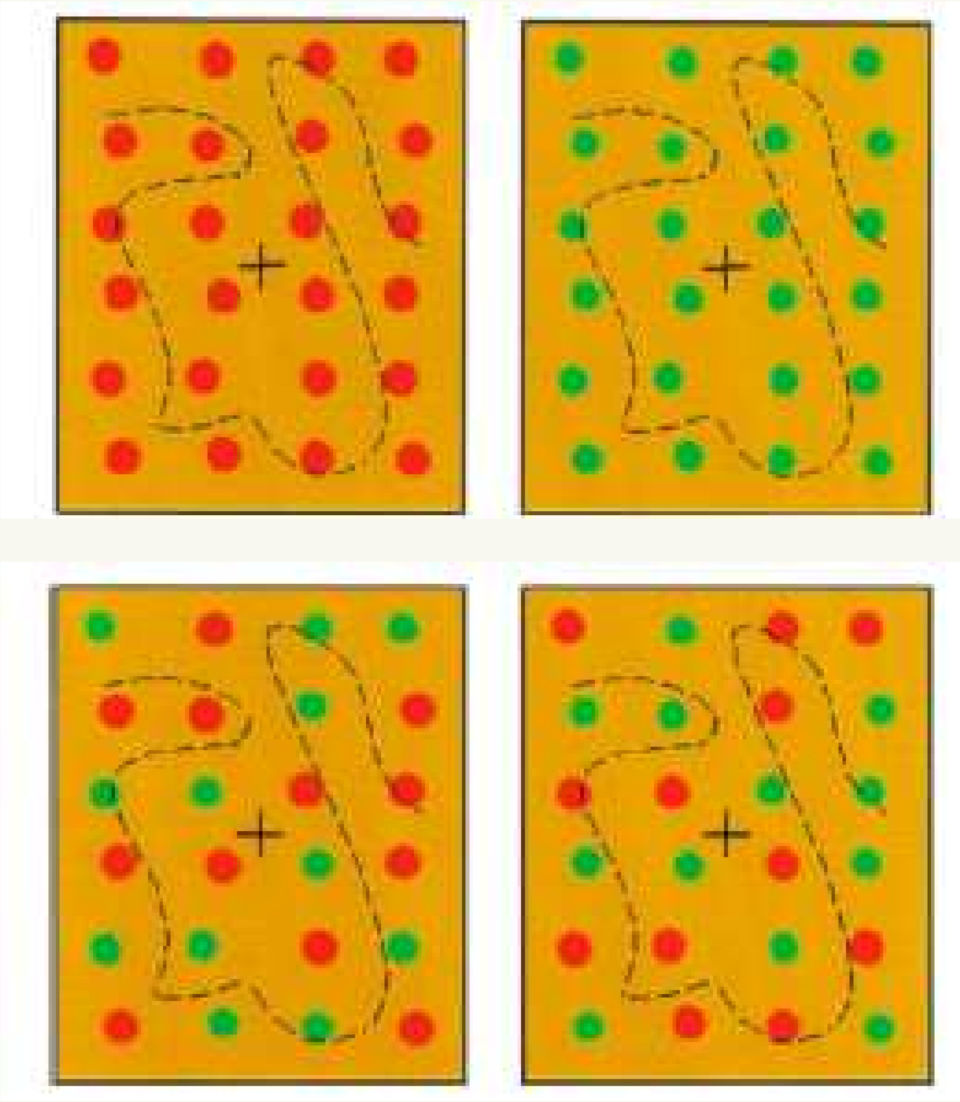


Kovács, Papathomas, Yang, and Fehér (1996)

Fourth Kovacs Experiment: Conventional Disc Patterns



Fourth Kovacs Experiment: Conventional Disc Patterns



Kovács, Papathomas, Yang, and Fehér (1996)

Tong's Simplified Rivalry Between Disc Patterns

- Rivalry between two **learned patterns**

Tong, Meng, and Blake (2006)

Tong’s Simplified Rivalry Between Disc Patterns

- Rivalry between two **learned patterns**

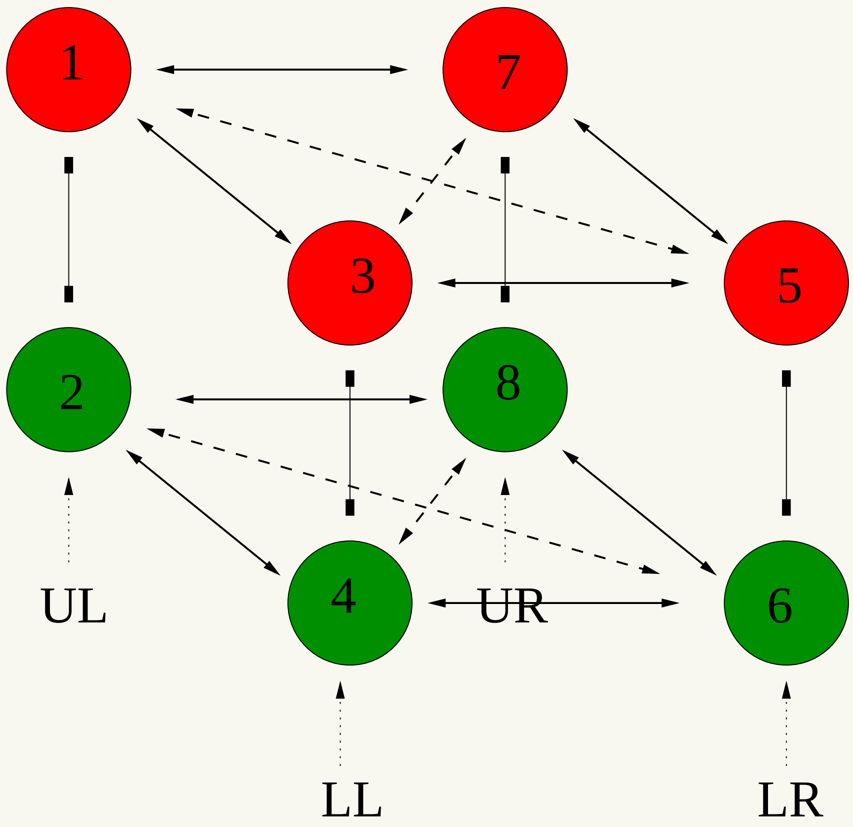
Tong, Meng, and Blake (2006)

- Rivalry between two **derived patterns** should also be observed

Rivalry Network for Conventional Tong Experiment

- Two learned patterns: RED and GREEN

  Symmetry group $\Gamma = \mathbf{D}_4 \times \mathbf{Z}_2(\rho)$



Diekman, G., and Wang (2013)

Patterns in Conventional Tong Experiment (1)

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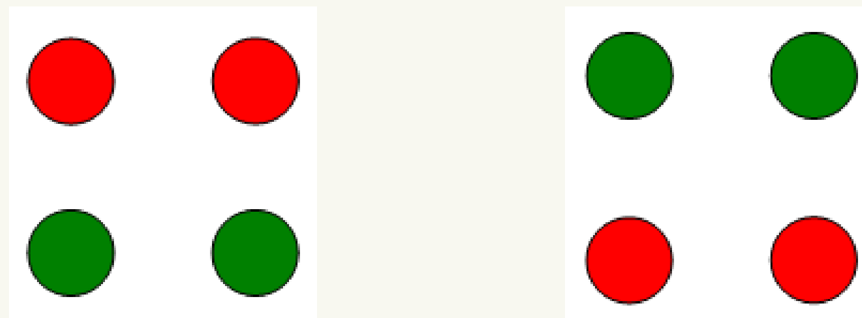
- Rivalry between **learned patterns**

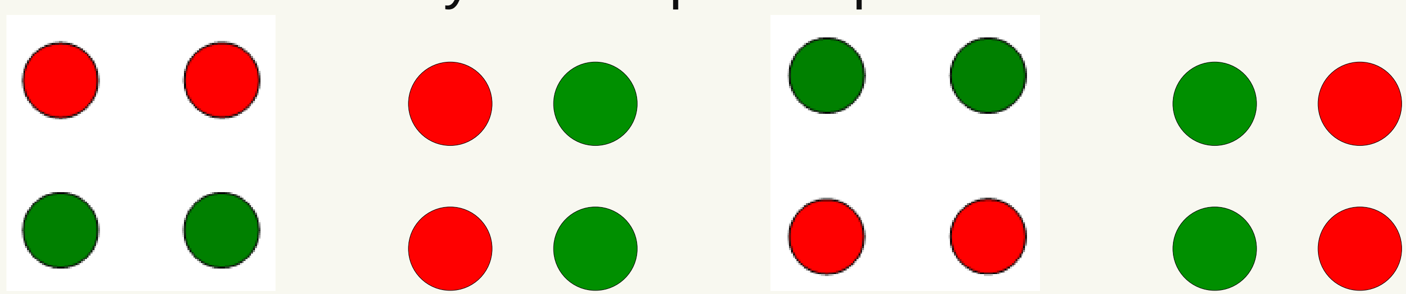
- Rivalry between **derived patterns**

Patterns in Conventional Tong Experiment (2)

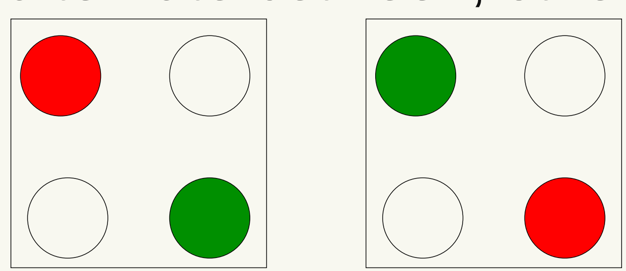
- Adjacent colors



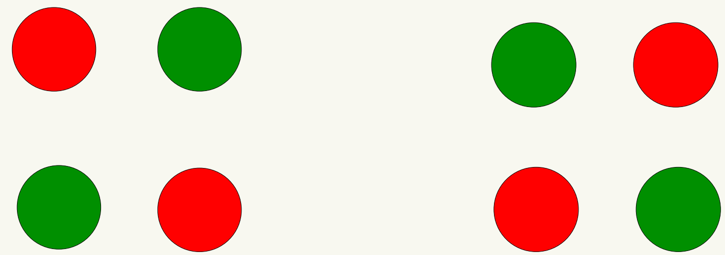
- Adjacent colors rotate by 90° in quarter period



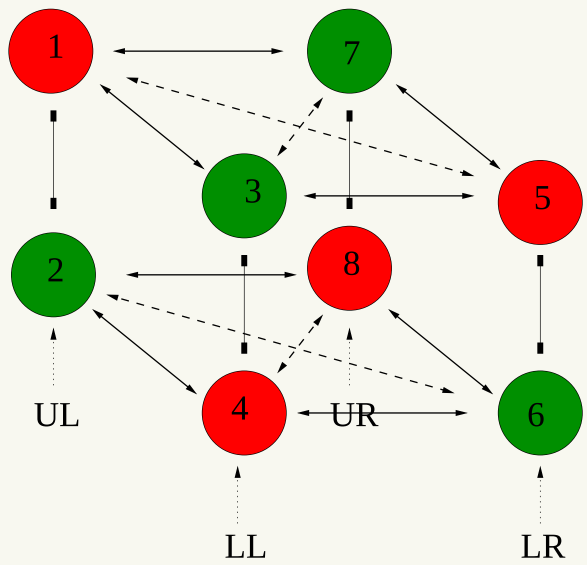
- Two dots on diagonal alternate between; other two dots are fused



Wilson Network for Scrambled Tong Experiment



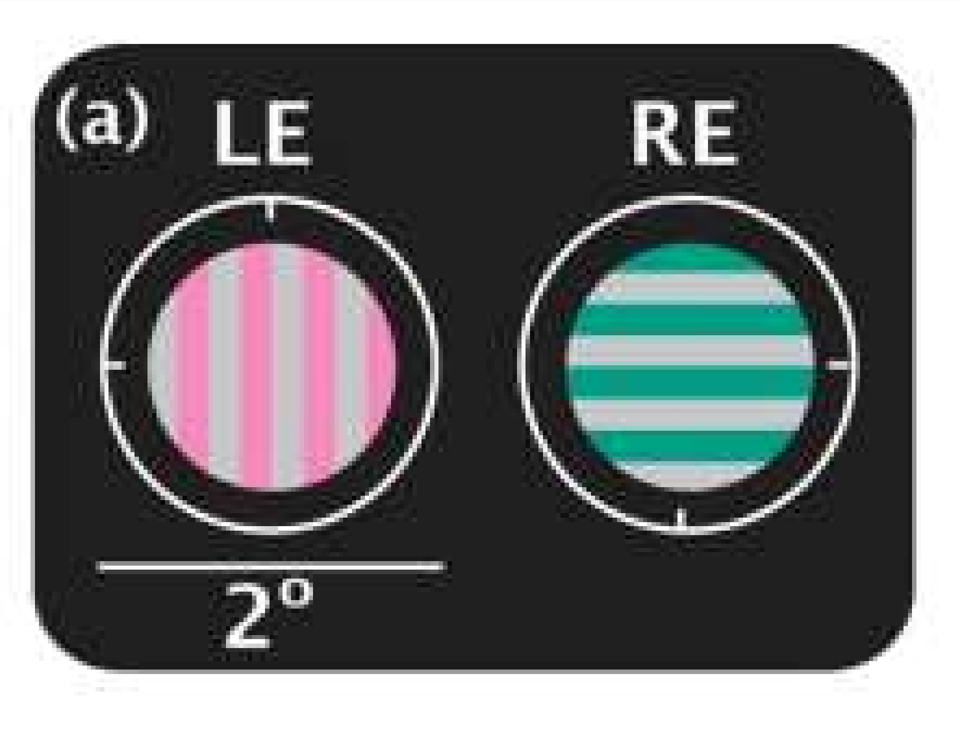
Learned Images in simplification of scrambled *colored dot* experiment



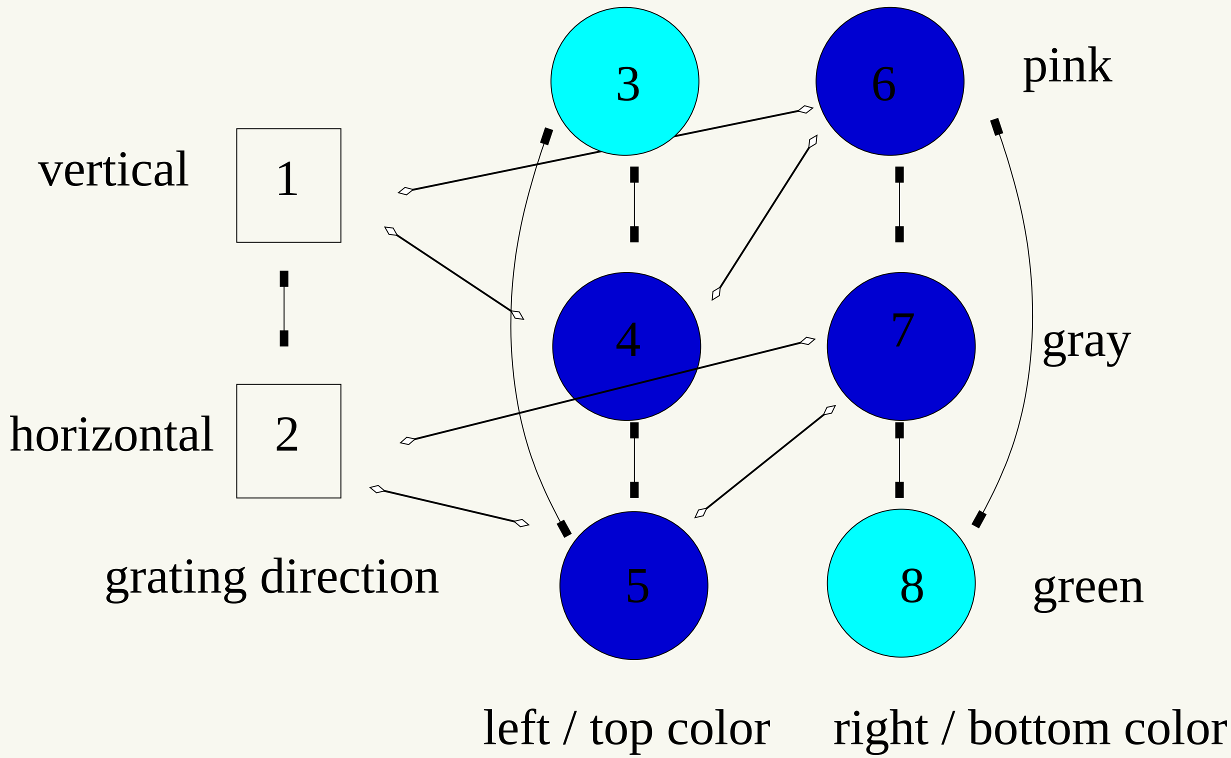
Network corresponding to simplified scrambled experiment

- Symmetry group is $\mathbf{D}_4 \times \mathbf{Z}_2(\rho)$
  Learned & derived **interchanged** with conventional experiment

Shevell, St. Clair, and Hong (2008)



Wilson Network for Shevell, St. Clair, and Hong



Symmetries

$$\begin{aligned} \rho &= (1\ 2)(4\ 5)(6\ 7) \\ \tau &= (3\ 8)(4\ 6)(5\ 7) \\ \rho\tau &= (1\ 2)(3\ 8)(4\ 7)(5\ 6) \end{aligned}$$

$\rho = -1 = \rho\tau$; $\tau = +1$

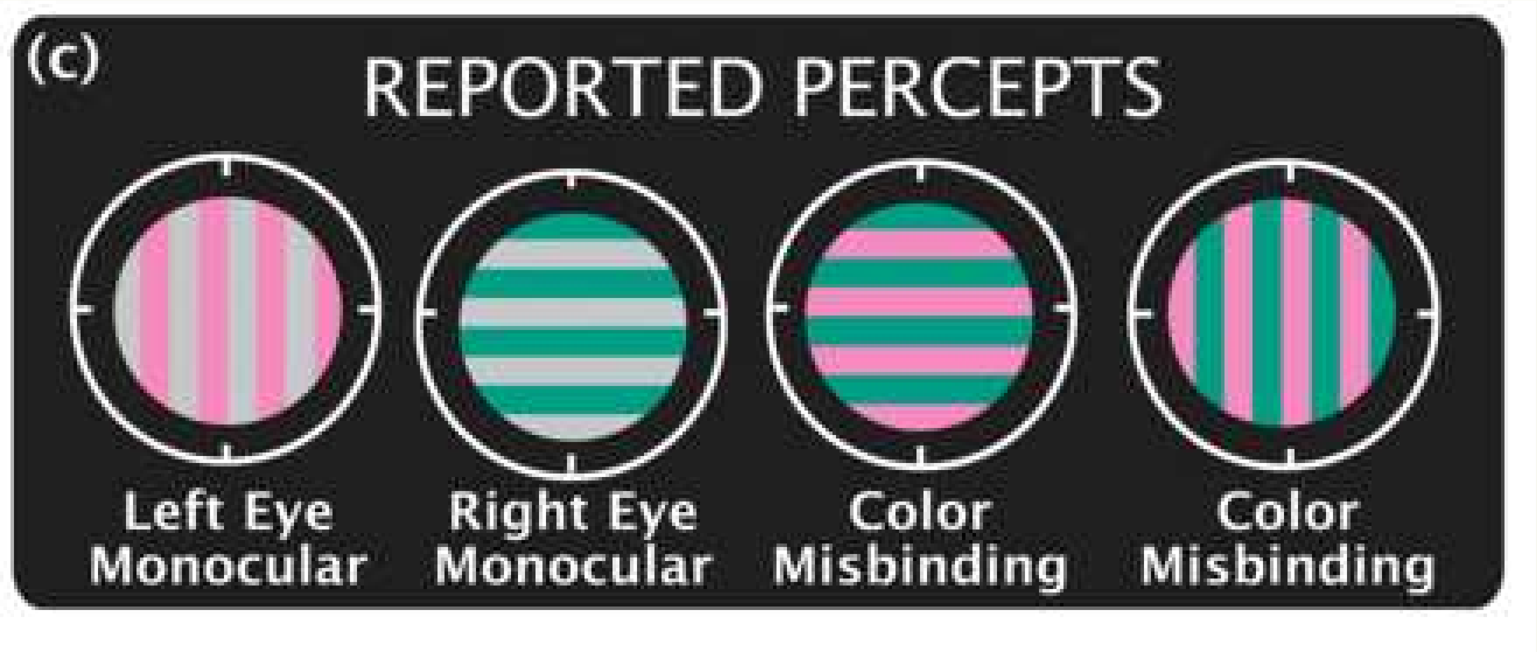
Fusion States

Gray-Pink $\leftrightarrow$ Green-Gray if $x_3^E < x_4^E = x_5^E$

Pink-Green if $x_3^E > x_4^E = x_5^E$

Maximally fused states $= \{x_1 = x_2; x_3 = x_8; x_4 = x_5 = x_6 = x_7\}$

Shevell, St. Clair & Hong 2008 Observed Percepts



Percepts observers reported during experiments

Synchrony and Synchrony Subspaces

RIGID PHASE-SHIFT SYNCHRONY

- $X(t) = (x_1(t), \ldots, x_n(t))$ exhibits **partial synchrony** if

$$x_c(t) = x_d(t)$$

for two nodes c and d.

- **polydiagonal** $=$ subspace

$$\Delta = \{x : x_c = x_d \quad \text{for some subset of pairs of cells} \quad c, d\}$$

- **synchrony subspace** $=$ polydiagonal flow-invariant $\forall$ admissibles

All solutions in a synchrony subspace exhibit partial synchrony

Synchrony Subspaces from Network Symmetry

- σ is a symmetry of $\dot{X} = F(X)$ if it maps solutions to solutions

  Equivalent to $F(\sigma x) = \sigma F(x)$

- $\mathrm{Fix}(\Sigma) = \{x \in \mathbf{R}^n : \sigma x = x \quad \forall \sigma \in \Sigma\}$ is flow-invariant

  Proof: $F(x) = F(\sigma x) = \sigma F(x)$

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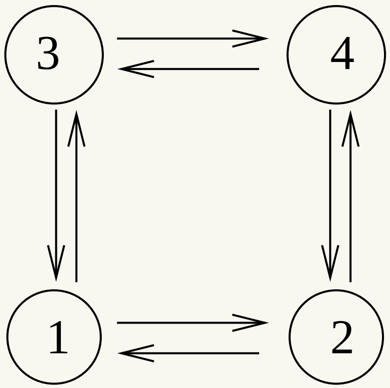


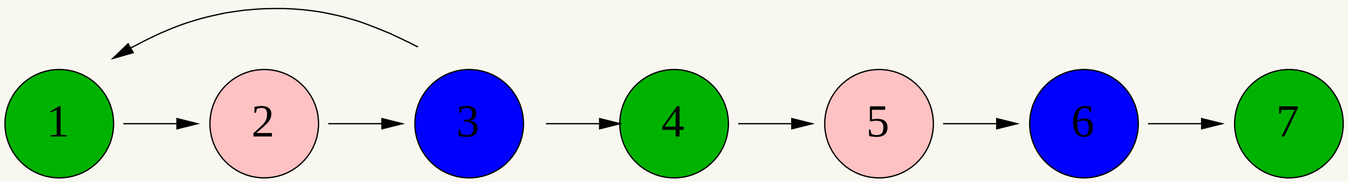
- If σ is permutation, then $\mathrm{Fix}(\sigma)$ is polydiagonal

  Example: $\mathrm{Fix}((23)(14)) = \{x_2 = x_3; x_1 = x_4\}$

- If Σ is subgroup of permutation symmetries, then $\mathrm{Fix}(\Sigma)$ is a synchrony subspace

Balanced Colorings

- Synchrony subspaces **do not** have to be fixed-point subspaces



$$\begin{aligned} \dot{x}_1 &= f(x_1, x_3) & \dot{x}_2 &= f(x_2, x_1) & \dot{x}_3 &= f(x_3, x_2) \\ \dot{x}_4 &= f(x_4, x_3) & \dot{x}_5 &= f(x_5, x_4) & \dot{x}_6 &= f(x_6, x_5) \\ \dot{x}_7 &= f(x_7, x_6) \end{aligned}$$

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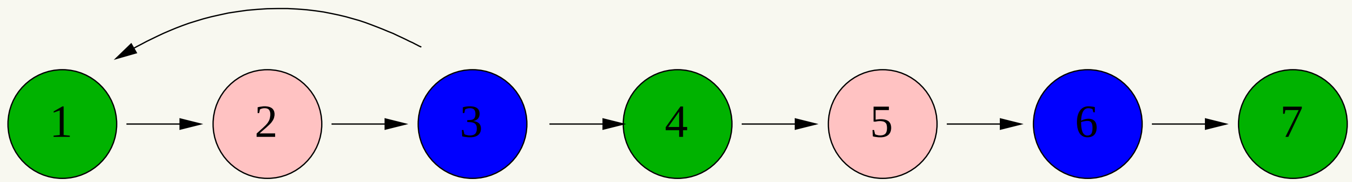

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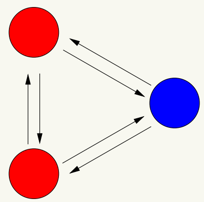
- $\Delta = \{x : x_1 = x_4 = x_7;\ x_2 = x_5;\ x_3 = x_6\}$ is flow-invariant
- Color cells the same color if cell coord's in polydiagonal Δ are equal
- Coloring is **balanced** if all cells with same color receive equal number of inputs from cells of a given color
- **Theorem 1**: **synchrony subspace $\Longleftrightarrow$ balanced**

  Stewart, G., and Pivato (2003); G., Stewart, and Török (2005)

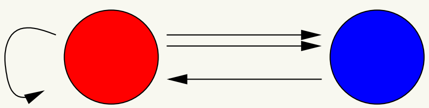
Quotient Networks with Self-Coupling & Multiarrows



- Synchrony subspace $\mathrm{Fix}(1\ 2)$ in bidirectional ring

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_3) \\ \dot{x}_2 &= f(x_2, x_3, x_1) \\ \dot{x}_3 &= f(x_3, x_1, x_2) \end{aligned} \qquad \text{where } f(x,y,z) = f(x,z,y)$$

- **Quotient network**:



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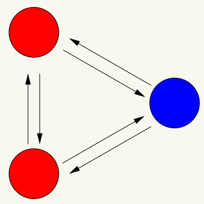
Quotient Networks with Self-Coupling & Multiarrows



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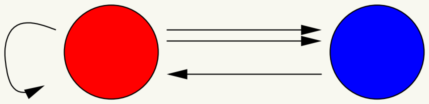

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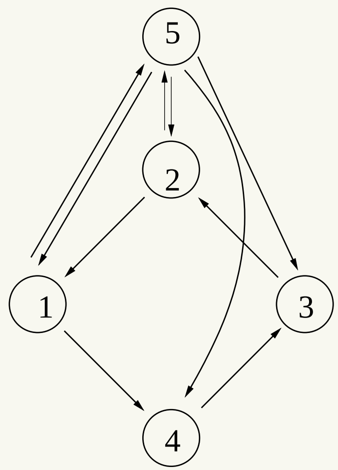
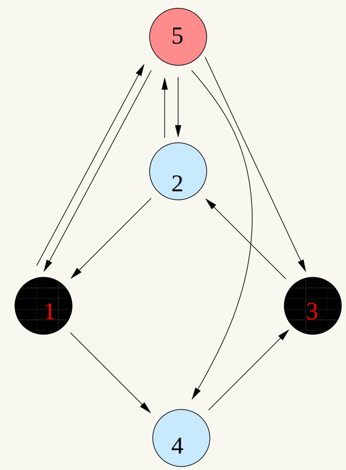
- Cell network with synchrony subspace leads to **quotient network**
- **Theorem 2**: Admissible ODE restricts to quotient admissible ODE

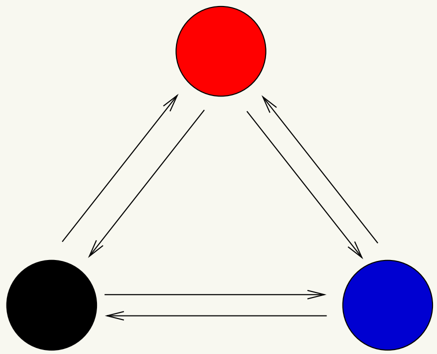
  Quotient admissible ODE lifts to admissible ODE

G., Stewart, and Török (2005)

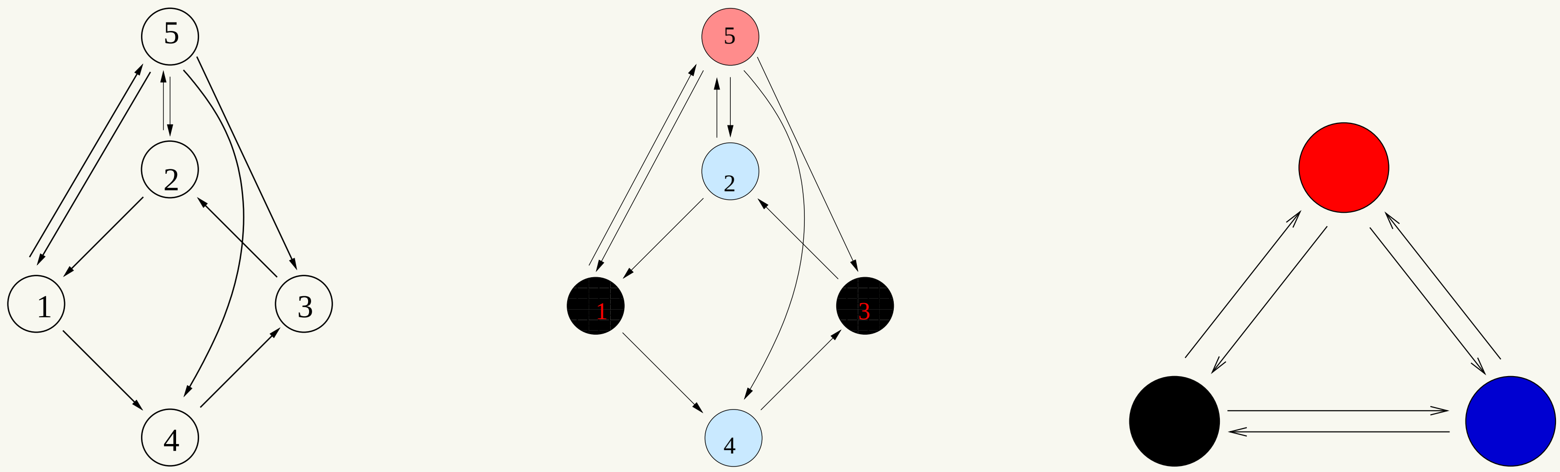
Asymmetric Network; Symmetric Quotient



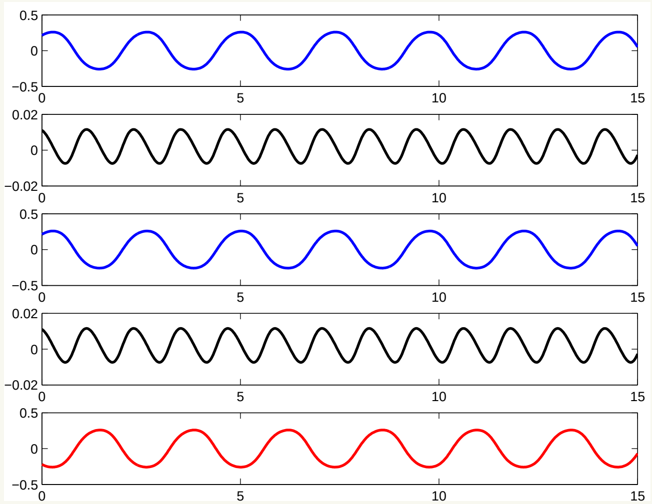




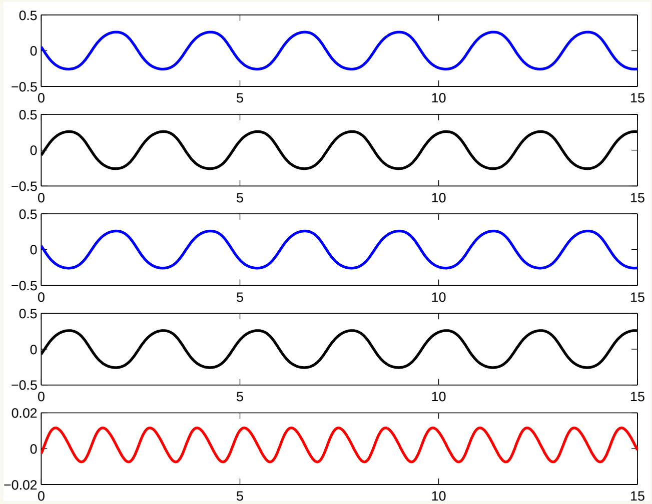
Asymmetric Network; Symmetric Quotient



- Quotient is bidirectional 3-cell ring with $\mathbf{S}_3$ symmetry



swap red and blue



swap blue and black

- **Rigid phase shift; no symmetry**

Rigid Phase-Shift $\Leftrightarrow$ Pattern of Phase-Shift Synchrony

Theorem 3: Assume network is **transitive**.

Nonzero **rigid** phase-shift synchrony
if and only if
phase-shift forced by **symmetry on quotient network**

- Let $Z(t)$ be the periodic solution.
  Quotient network corresponds to the synchrony subspace

$$\Delta_Z = \{x_i = x_j \quad \text{if} \quad z_i(t) = z_j(t)\ \forall t\}$$

Stewart and Parker (2008, 2009); G., Romano and Wang (2010, 2011)

Thanks

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Jim Collins	*Boston University*	Gaits
Ian Stewart	Warwick	Gaits, Phase-Shift Synchrony
David Romano	*Grinnell*	Phase-Shift Synchrony
Yunjiao Wang	Texas Southern	Phase-Shift Synchrony, Rivalry
Tyler McMillen	Cal State Fullerton	Rivalry
Casey Diekman	NJIT	Rivalry

Wilson Networks

Coupled cell networks have nodes and arrows $\&$ equivalence classes of nodes and arrows. **Wilson network** = coupled cell network such that

(a) Nodes partition into **attribute columns**: $\mathcal{C} = A_1 \cup \cdots \cup A_m$ where all nodes in an attribute column are cell equivalent.
Example: Attribute might be color of a dot

(b) **Pattern =** choice of one node in each attribute column
Learned patterns given; other patterns **derived**
Binocular rivalry has two learned patterns

(c) Two types of arrows: *inhibitory* and *excitatory*

- Nodes in same attribute column connected by inhibitory arrow
- Nodes in same learned pattern connected by excitatory arrow
- Inhibitory arrows and excitatory arrows are not arrow equivalent

Wilson considers networks where nodes are cell equivalent, inhibitory arrows are arrow equivalent, and excitatory arrows are arrow equivalent

Rivalry Networks

A **rivalry network** is a Wilson network such that

(a) Attributes partition into **attribute types**
Nodes in attribute equivalent columns are cell equivalent
Inhibitory arrows are equivalent iff have equivalent attributes

(b) **Feature** is property of pairs of nodes in different attribute columns
Excitatory arrows equivalent iff connect nodes with same features

Examples:

- Dot experiments have *distance feature*: assigns to pair of nodes distance between dots
  Arrows can be equivalent only if connect equidistant nodes
- Dot experiments have *color feature*
  Excitatory arrows connecting nodes of same color and arrows connecting nodes of different colors are not equivalent