Introduction to Nonlinear Dynamics

Santa Fe Institute

Complex Systems Winter School

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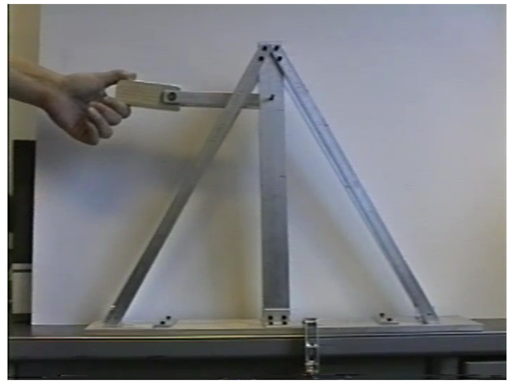
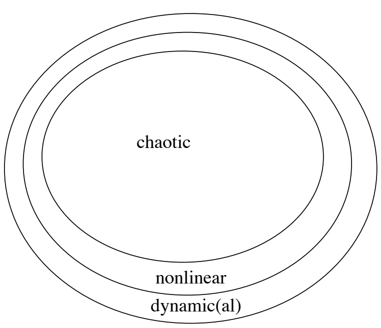


http://ayresriverblog.com

Chaos

Complex behavior, arising in a deterministic nonlinear dynamic system, which exhibits two special properties:

- sensitive dependence on initial conditions
- characteristic structure…



Chaos

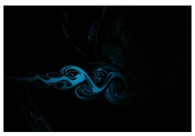
Complex behavior, arising in a deterministic nonlinear dynamic system, which exhibits two special properties:

- sensitive dependence on initial conditions
- characteristic structure…

Systems that exhibit chaos are ubiquitous; many of them are also simple, well-known, and "well-understood"

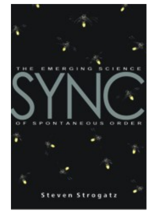
Where nonlinear dynamics turns up

- Flows (of fluids, heat, …)
  - Eddy in creek
  - Weather
  - Vortices around marine invertebrates
  - Air/fuel flow in combustion chambers



Where nonlinear dynamics turns up

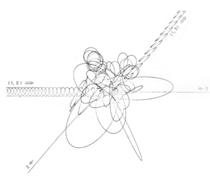
- Driven nonlinear oscillators
  - Pendula
  - Hearts
  - Fireflies



  - and lots of other electronic, chemical, & biological systems

Where nonlinear dynamics turns up

- Classical mechanics
  - three-body problem
  - paired black holes
  - pulsar emission
  - ….



Hut & Bahcall *Ap.J.* ***268****:319*

- Protein folding
- Population biology
- And many, many other fields (including yours)

- discrete time systems:
  - time proceeds in clicks
  - “maps”
  - modeling tool: difference equation
- continuous time systems:
  - time proceeds smoothly
  - “flows”
  - modeling tool: differential equations

A useful graphical solution technique

- "cobweb" diagram
- *aka* return map
- *aka* correlation plot

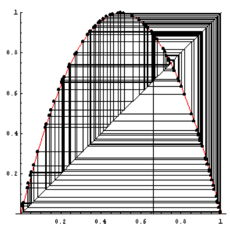
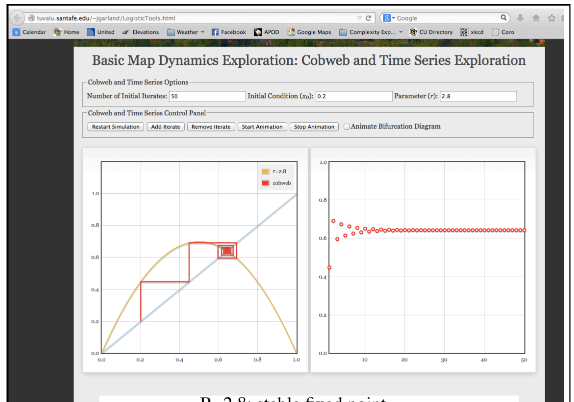


Image from Doug Ravenel's website at URochester

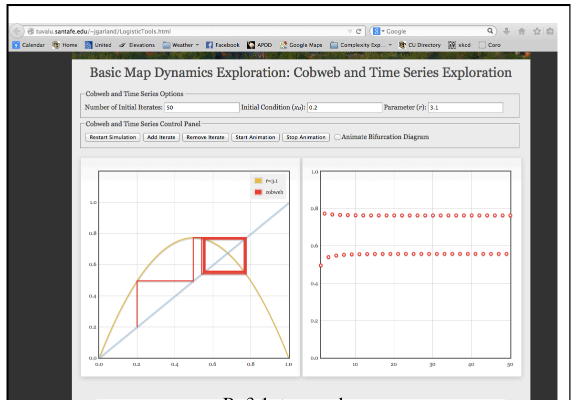


R=2.8: stable fixed point

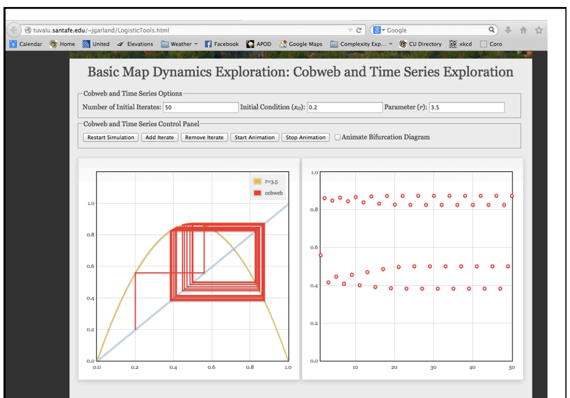
Bifurcations

Qualitative changes in the dynamics caused by changes in ***parameters:***

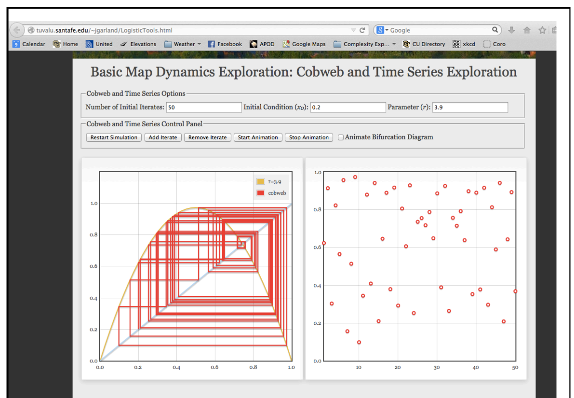
- Heart: pathology
- Eddy in creek: water level
- Olfactory bulb: smell
- Brain: blood chemicals
- Logistic map: R parameter…



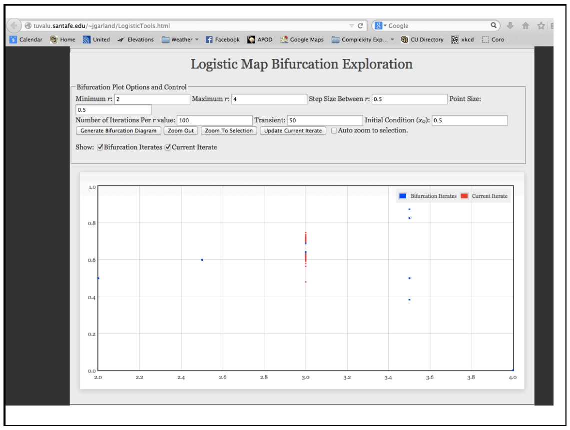
R=3.1: two cycle

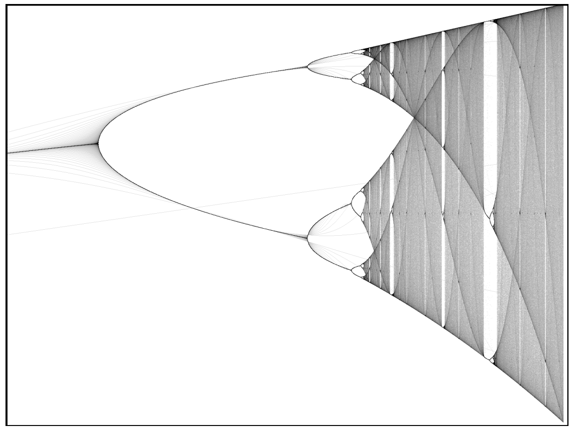


R=3.5: four cycle

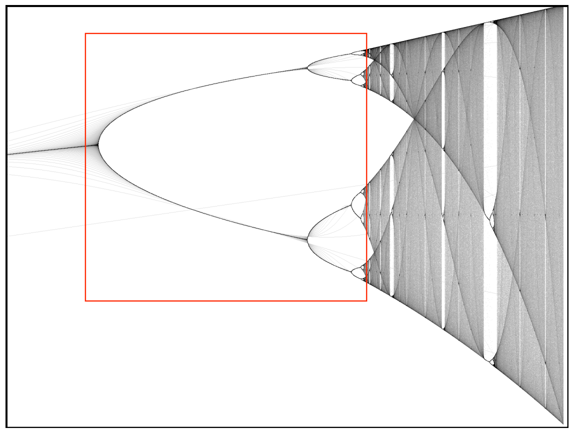


R=3.9: chaos



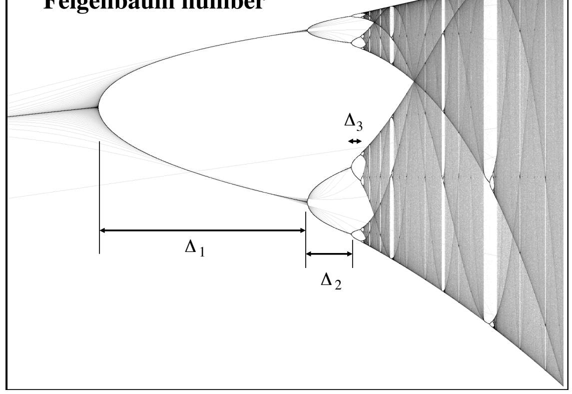


- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)



- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- *period-doubling cascade @ low R*

Feigenbaum number

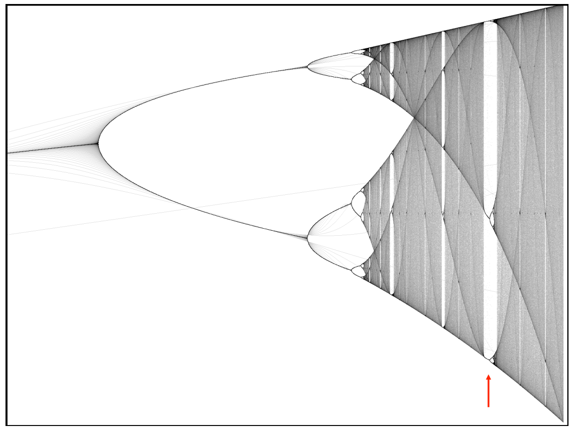


Universality!

Feigenbaum number and many other interesting chaotic/dynamical properties hold *for any 1D map with a quadratic maximum*.

Proof: renormalizations. See Strogatz §10.7

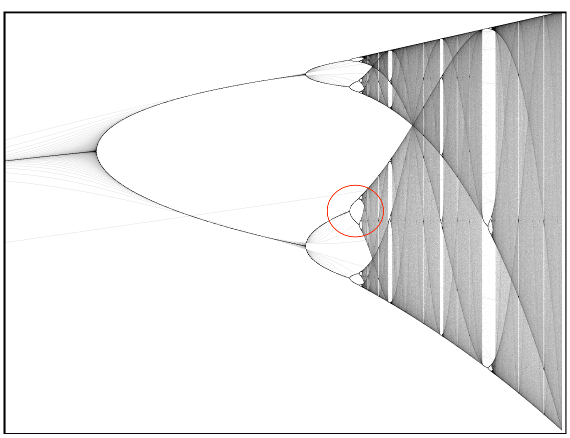
Don't take this too far, though…



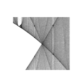
- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- period-doubling cascade @ low R
- *windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)*

There's something very special about 3…

- Sarkovskii (1964)
  3, 5, 7, …3x2, 5x2, …3x2^2, 5x2^2, … 2^2, 2, 1
- Yorke (1975)
- Metropolis *et al.* (1973)

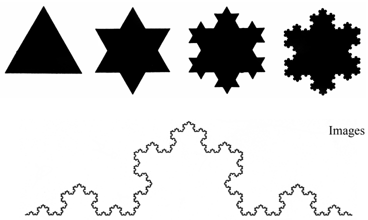


- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- period-doubling cascade @ low R
- windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)
- *small copies of object embedded in it (fractal)*

(lots of other interesting stuff, too — e.g., Misiurewicz points) 

Fractals

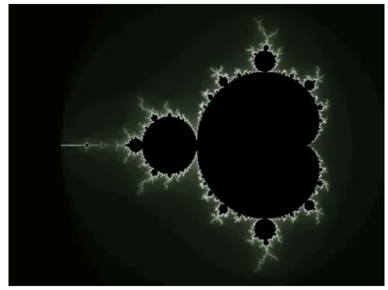
- non-integer Hausdorff dimension
- self-similar



Images from Gleick

Canonical example: the Cantor set!

Another canonical example: the Mandelbrot set



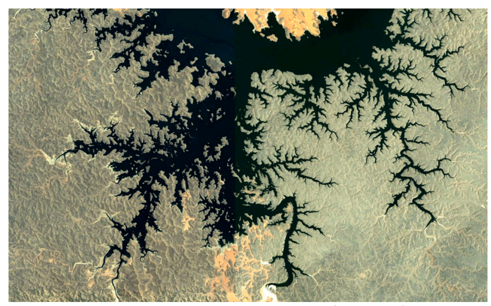
www.youtube.com/watch?v=G_GBwuYuOOs

Fractals basins and basin boundaries

Newton's method on $x^4 - 1 = 0$

From Strogatz

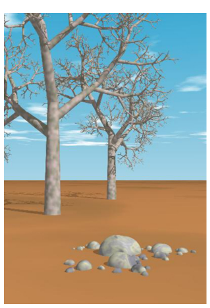
Fractals in the wild



paulbourke.net/fractals/googleearth/

See also: coastlines, trees, lungs, clouds, snowflakes …

Fractals in computer graphics



Matthew Ward, WPI
davis.wpi.edu/~matt/courses/fractals/trees.html

Fractals and chaos…

The connection: *many (most)* chaotic systems have fractal state-space structure.

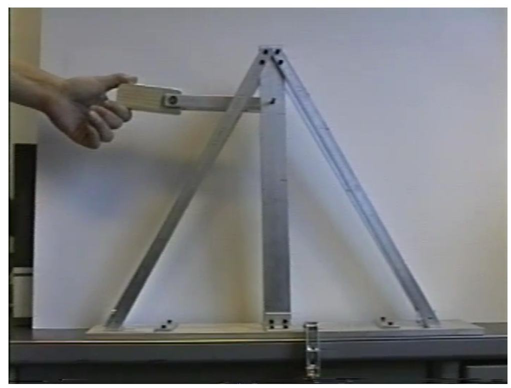
But **not** "*all*."

So far: mostly about *maps.*

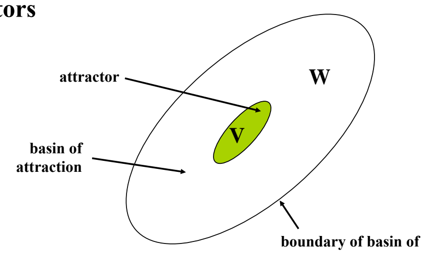
- discrete time systems:
  - time proceeds in clicks
  - "maps"
  - modeling tool: differ***ence*** equation

Next up: *flows*

- continuous time systems:
  - time proceeds smoothly
  - "flows"
  - modeling tool: differ***ential*** equations



Attractors



- Attractors exist only in dissipative systems!
- Dissipation ⟺ contraction of state space under the influence of the dynamics
- Can still have chaos if no dissipation…just not chaotic *attractors*

Courtesy of Allison Brown
Best Poster prize, Experimental Chaos Conference, 2012

Conditions for chaos in continuous-time systems

Necessary:

- Nonlinear
- At least three state-space dimensions (NB: only one needed in maps)

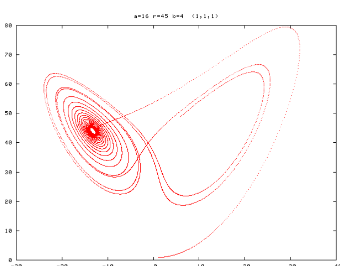
Necessary and sufficient:

- "Nonintegrable"

  i.e., cannot be solved in closed form

Concepts: review

- State variable
- State space
- Initial condition
- Trajectory
- Attractor
- Basin of attraction
- Transient
- Fixed point (un/stable)
- Bifurcation
- Parameter



Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

ABSTRACT

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

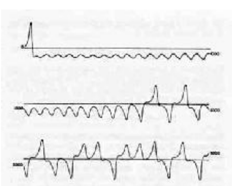


FIG. 1. Numerical solution of the convection equations. Graph of Y as a function of time for the first 1000 iterations (upper curve), second 1000 iterations (middle curve), and third 1000 iterations (lower curve).

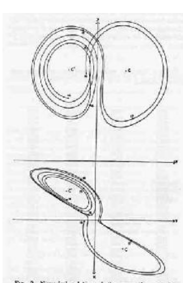


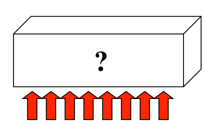
FIG. 2. Numerical solution of the convection equations. Projections on the X-Y-plane and the Y-Z-plane in phase space of the segment of the trajectory extending from iteration 1400 to iteration 1900. Numerals "14," "15," etc., denote positions at iterations 1400, 1500, etc. States of steady convection are denoted by C and C'.

- Equations:

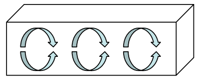
$x' = a(y - x)$

$y' = rx - y - xz$

$z' = xy - bz$

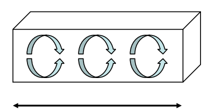


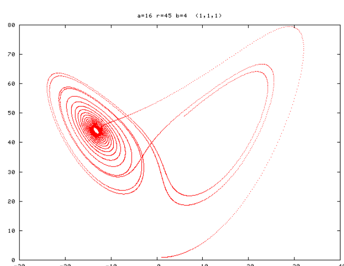
(first three terms of a Fourier expansion of the Navier-Stokes eqns)



- State variables:
  - *x* convective intensity
  - *y* temperature
  - *z* deviation from linearity in the vertical convection profile

- Parameters:
  - *a* Prandtl number - fluids property
  - *r* Rayleigh number - related to ΔT
  - *b* aspect ratio of the fluid sheet

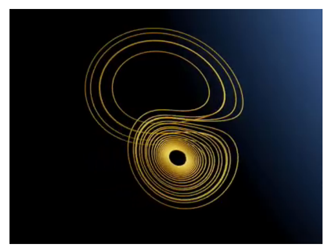


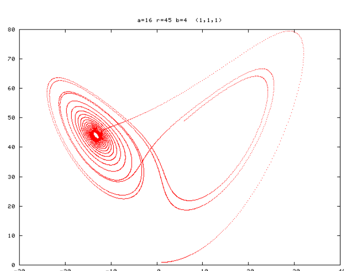


$x' = 16(y - x)$

$y' = 45x - y - xz$

$z' = xy - 4z$





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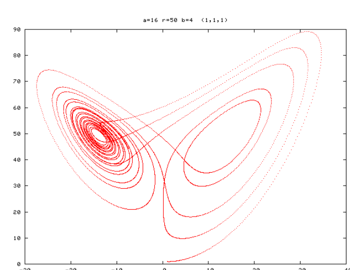
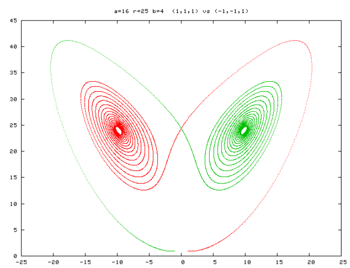
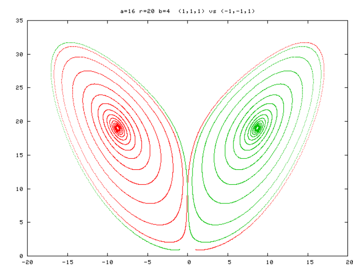


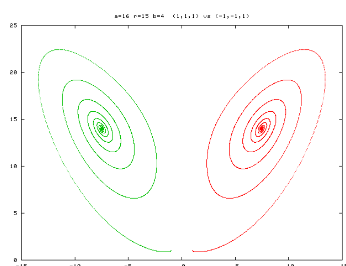
$x' = 16(y - x)$

$y' = \mathbf{50}x - y - xz$

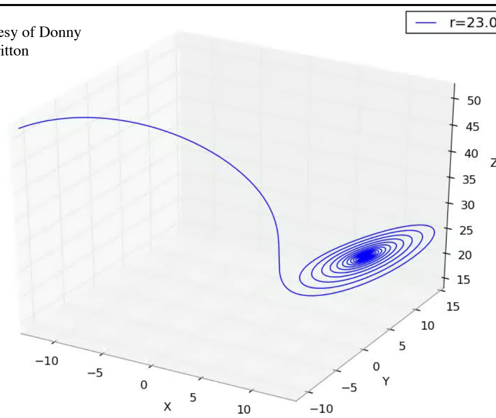
$z' = xy - 4z$







Courtesy of Donny Warbritton



Before we leave Lorenz…

Deterministic Nonperiodic Flow[1]

Edward N. Lorenz

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Abstract

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

Attractors

Four types:

- fixed points
- limit cycles (*aka* periodic orbits)
- quasiperiodic orbits
- chaotic attractors

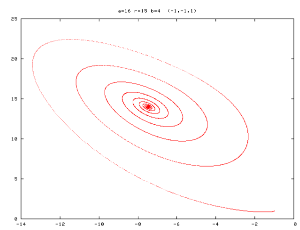
A nonlinear system can have any number of attractors, of all types, sprinkled around its state space

Their basins of attraction (plus the basin boundaries) *partition* the state space

And there's no way, *a priori*, to know where they are, how many there are, what types, etc.

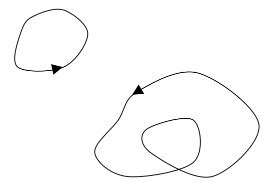
Attractors

- Fixed point



Attractors

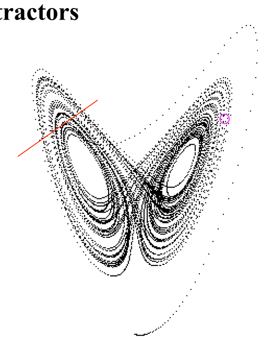
- Limit cycle



Attractors

- Quasi-periodic orbit…

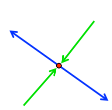
"Strange" or chaotic attractors



- *often* fractal
- covered densely by trajectories
- exponential divergence of neighboring trajectories…

Lyapunov exponents

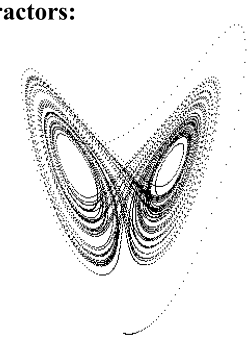
- nonlinear analogs of eigenvalues: one λ for each dimension



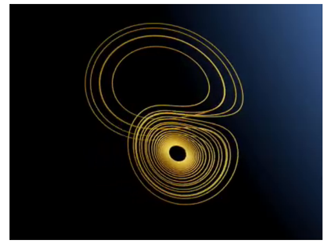
Lyapunov exponents: summary

- nonlinear analogs of eigenvalues: one λ for each dimension
- negative λ_i compress state space; positive λ_i stretch it
- $\Sigma\lambda_i < 0$ for dissipative systems
- λ_i are same for all ICs in one basin
- long-term average in definition; biggest one (λ_1) dominates as t → ∞
- *positive* λ_1 *is a signature of chaos*

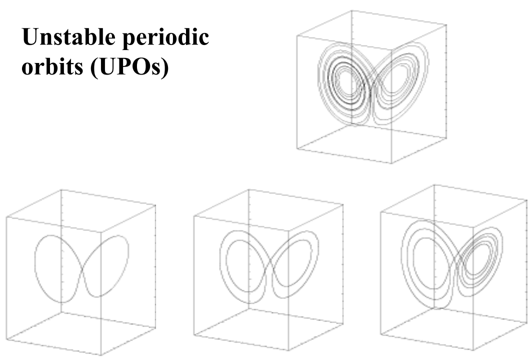
"Strange" or chaotic attractors:



- exponential divergence of neighboring trajectories
- *often* fractal
- covered densely by trajectories
- contain an infinite number of "unstable periodic orbits"…

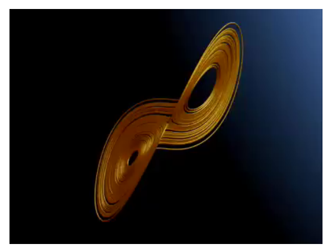


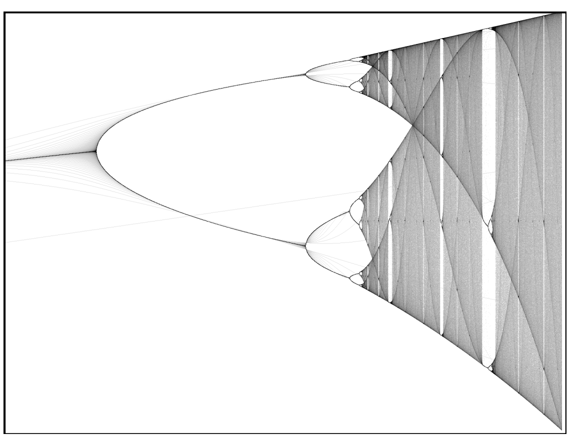
Unstable periodic orbits (UPOs)



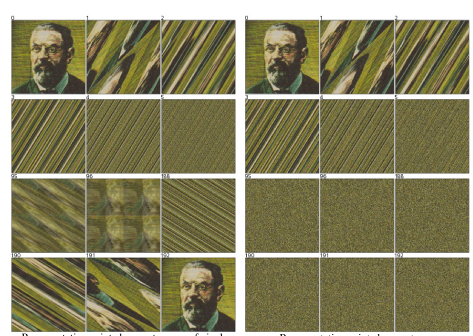
Bradley/Mantilla, *Chaos* **12**:596

"Attractor bones"…





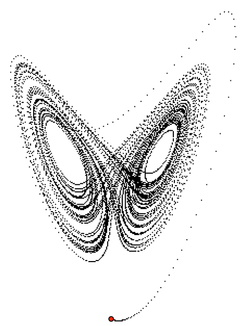
Poincare recurrence

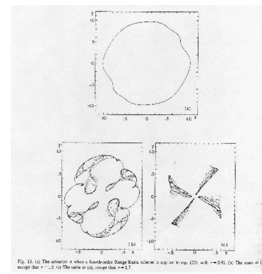


Representative point chosen at corner of pixel

Representative point chosen at a random place in pixel

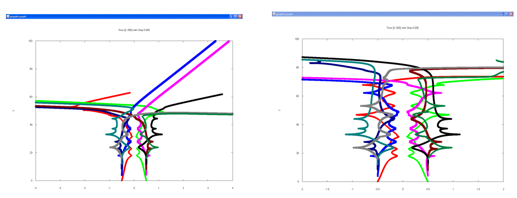
Crutchfield *et al*. *Chaos* **255**:46 and
http://www.mpipks-dresden.mpg.de/mpi-doc/kantzgruppe/wiki/projects/Recurrence.html





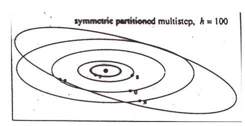
Different timestep

Lorenz, *Physica D* **35:**229

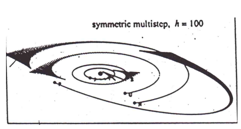


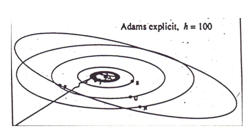
Different arithmetic

N. Ross Ph.D. thesis, Ucolorado, 2008



Different solver algorithm…





Moral: numerical methods can run amok in "interesting" ways…

- can cause distortions, bifurcations, etc.
- and these look a lot like *real, physical* dynamics…
- source: algorithms, arithmetic system, timestep, etc.
- Q: what could you do to diagnose whether your results included spurious numerical dynamics?

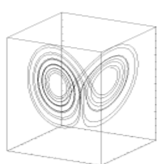
Moral: numerical methods can run amok in "interesting" ways…

- can cause distortions, bifurcations, etc.
- and these look a lot like *real, physical* dynamics…
- source: algorithms, arithmetic system, timestep, etc.
- Q: what could you do to diagnose whether your results included spurious numerical dynamics?
  - *change the timestep*
  - *change the method*
  - *change the arithmetic*

But beware machine ε…

So ODE solvers make mistakes.

…and chaotic systems are sensitively dependent on initial conditions….



…??!?

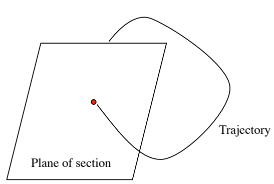
Shadowing lemma

Every* noise-added trajectory on a chaotic attractor is *shadowed* by a true trajectory.

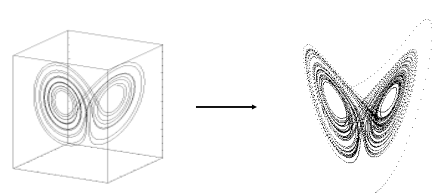
Important: this is for *state* noise, not *parameter* noise.

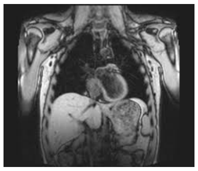
(*) Caveat: not if the noise bumps the trajectory out of the basin

Section

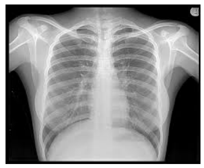


Not the same thing as a projection!



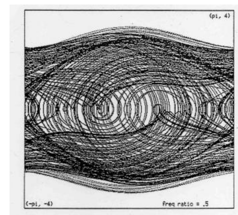


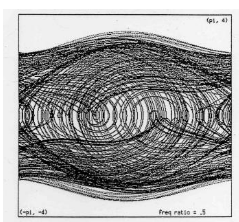
mri.radiology.uiowa.edu



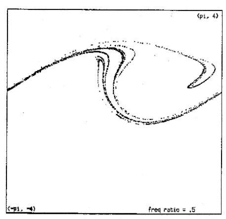
prixray.com

The driven damped pendulum





trajectory

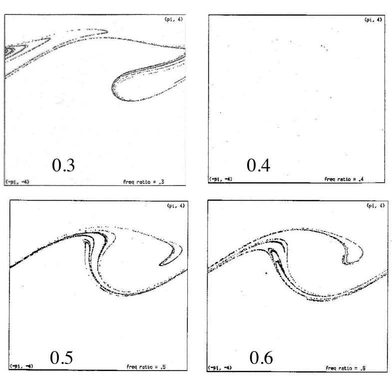


Poincare section

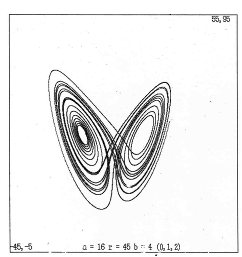
Time-slice sections of periodic orbits: some thought experiments

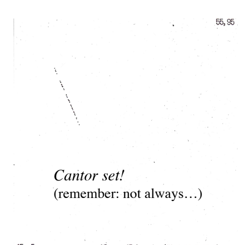
- pendulum rotating @ 1 Hz and strobe @ 1 Hz?
- pendulum rotating @ 1 Hz and strobe @ 2 Hz?
- pendulum rotating @ 1 Hz and strobe @ 3 Hz?
- pendulum rotating @ 1 Hz and strobe @ 1/2 Hz?
- pendulum rotating @ 1 Hz and strobe @ π Hz? (or some other irrational)

What bifurcations look like on a Poincare section

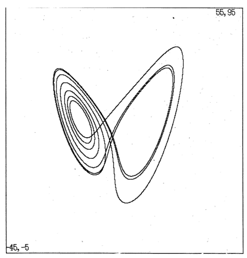


Spatial sections

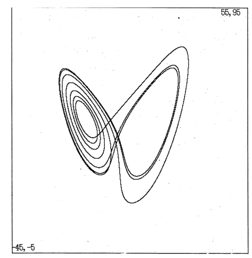


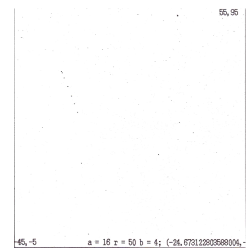


What about a section of a UPO?

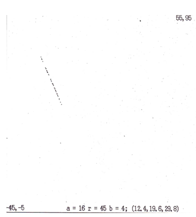


?





Aside: finding UPOs

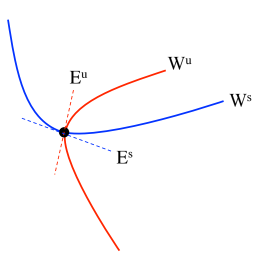


- Section
- Look for close returns
- Cluster
- Average
- See Gunaratne, So papers

Computing sections

- If you're slicing in state space: use the "inside-outside" function
- If you're slicing in *time*: use modulo on the timestamp
- See Parker & Chua for more details

λ_i and the un/stable manifolds (W^u and W^s)

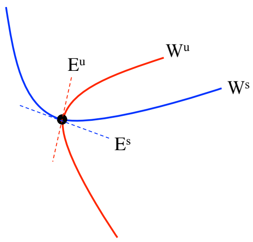


Aside: finding those un/stable manifolds

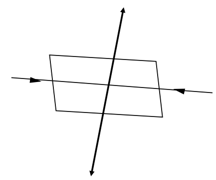
- Linearize the system
- Find the eigenvectors E^s and E^u
- Take a step along E^s; run time forwards
- Take a step along E^u; run time backwards
- See Osinga & Krauskopf paper for more details

Note: saddles are not the only possible landscape geometry around fixed points (they're just the most interesting ones!)

These λ_i & manifolds play a critical role in the control of chaos…



Local-linear control* of a hyperbolic point



* e.g., via pole placement

Lyapunov exponents, revisited:

- n-dim system has n λ_i; $\Sigma\lambda_i < 0$ for dissipative systems
- λ_i are same for all ICs in one basin
- negative λ_i compress state space along *stable manifolds*
- positive λ_i stretch it along *unstable manifolds*
- biggest one (λ_1) dominates as $t \to \infty$
- positive λ_1 is a signature of chaos
- calculating them:
  - From equations: eigenvalues of the variational matrix (see variational system notes on CSCI5446 course webpage, which you can access from Liz's homepage.)
  - From data: various creative algorithms…

Calculating λ (& other invariants) from data

- The bible: H. Kantz & T. Schreiber, *Nonlinear Time Series Analysis*
- Associated software: TISEAN www.mpipks-dresden.mpg.de/~tisean
- A recent review article: EB & H. Kantz, "Nonlinear Time Series Analysis Revisited," *CHAOS* **25:**097610 (2015)

TISEAN

Nonlinear Time Series Analysis

Rainer Hegger
Holger Kantz
Thomas Schreiber

Go to Version 3.0.1 (released March 2007)

Go to Version 2.1 (released December 2000)

TISEAN 3.0.1: Table of Contents

All programs in alphabetical order

Sections

- Generating time series
- Linear tools
- Utilities
- Stationarity
- Embedding and Poincaré sections
- Prediction
- Noise reduction
- Dimension and entropy estimation
- Lyapunov exponents
- Surrogate data
- Spike trains
- XTisean
- Unsupported

TISEAN 3.0.1

TISEAN home

Table of Contents

General Manual

Surrogates Manual

Tutorial

Usage Notes

Installation

Problems

Generating time series

A few routines are provided to generate test data from simple equations. Since there are powerfull packages (for ex
Helena Nusse and Jim Yorke) that can generate chaotic data, we have only included a minimal selection here.

Lyapunov exponents are an important means of quantification for unstable systems. They are however difficult to estimate from a time series. Unless low dimensional, high quality data is at hand, one should not attempt to calculate the full spectrum. Try to compute the maximal exponent first. The two implementations differ slightly. While lyap_k implements the formula by Kantz, lyap_r uses that by Rosenstein et al. which differs only in the definition of the neighbourhoods. We recommend to use the former version, lyap_k.

The estimation of Lyapunov exponents is also discussed in the introduction paper. A recent addition is a programm to compute finite time exponents which are not invariant but contain additional information.

Maximal exponent	lyap_k, lyap_r
Lyapunov spectrum	lyap_spec

Description of the program: lyap_k

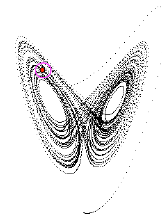
The program estimates the largest Lyapunov exponent of a given scalar data set using the algorithm of Kantz.

Usage:

lyap_k [Options]

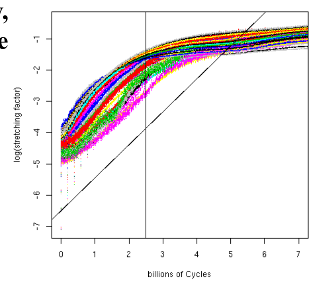
Everything not being a valid option will be interpreted as a potential datafile name. Given no datafile at all, means read stdin. Also - means stdin

Kantz's algorithm:



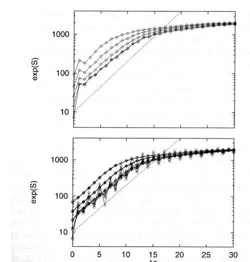
1. Choose point K •
2. Look at the points around it (ε neighborhood)
3. Measure how far they are from K
4. Average those distances
5. Watch how that average grows with time (Δn)
6. Take the log, normalize over time ➔ S(Δn)
7. Repeat for lots of points K and average the S(Δn)

If you're lucky, things look like this.



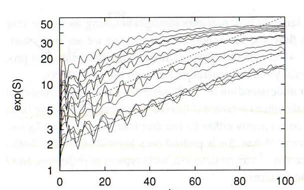
- The slope of the scaling region—iff one exists—is the λ

Or this:



This is fig 5.3 in Kantz & Schreiber

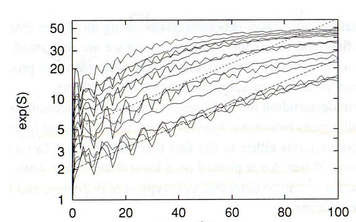
If you're not lucky:



- The slope of the scaling region—iff one exists—is the λ

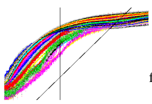
This is fig 5.4 in Kantz & Schreiber

What do you think those oscillations might be?



Calculating λ (& other invariants) from data

Be careful! TISEAN has lots of knobs and its results are incredibly sensitive to their values!



Different colors on that plot from before = different settings for one of those knobs

---

Option	Description	Default
-l#	number of data to be used	whole file
-x#	number of lines to be ignored	0
-c#	column to be read	1
-M#	maximal embedding dimension to use	2
-m#	minimal embedding dimension to use	2
-d#	delay to use	1
-r#	minimal length scale to search neighbors	(data interval)/1000
-R#	maximal length scale to search neighbors	(data interval)/100
-##	number of length scales to use	5
-n#	number of reference points to use	all
-s#	number of iterations in time	50
-t#	'theiler window'	0
-o#	output file name	without file name: 'datafile'.lyap (or stdin.lyap if the data were read from stdin)
-V#	verbosity level 0: only panic messages 1: add input/output messages 2: add statistics for each iteration	3
-h	show these options	none

Description of the Output:

For each embedding dimension and each length scale the file contains a block of data consisting of 3 columns

1. The number of the iteration
2. The logarithm of the stretching factor (the slope is the Lyapunov exponent if it is a straight line)
3. The number of points for which a neighborhood with enough points was found

---

Calculating λ (& other invariants) from data

- *Be careful! TISEAN has lots of knobs and its results are incredibly sensitive to their values!*

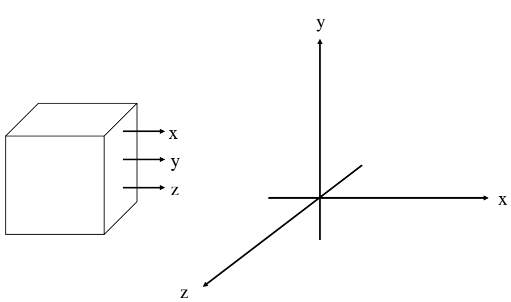
- Use your dynamics knowledge to understand & use those knobs intelligently

- *Look* at the results plots. For example, do not blindly fit a regression line to something that has no scaling region (this is a good idea in general, of course)

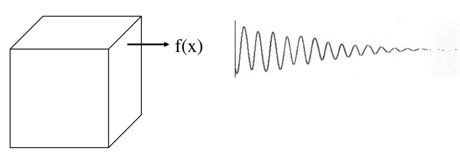
Fractal dimension:

- Capacity
- Box counting
- Correlation (d2 in TISEAN)
- Lots of others:
  - Kth nearest neighbor
  - Similarity
  - Information
  - Lyapunov
  - …
- See Chapter 6 and §11.3 of Kantz & Schreiber

We've been assuming that we can measure all the state variables…

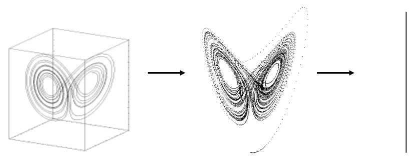


But often you can't.



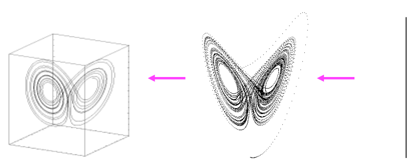
Rarely do you even *know* what they are. And even if you did, you might not be able to measure them directly...

How to undo a projection?

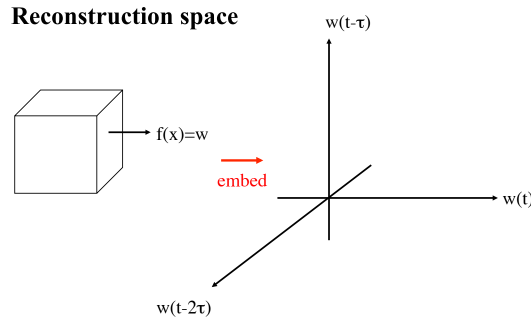


Delay-coordinate embedding

“reinflate” that squashed data to get a *topologically identical* copy of the original thing.

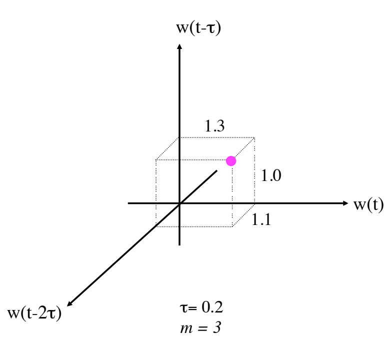


Reconstruction space

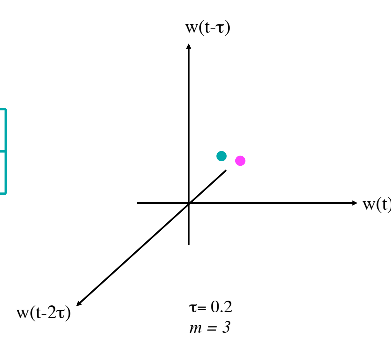


Mechanics

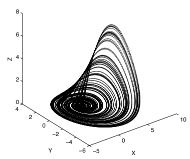
w	t
1.3	0.1
1.2	0.2
1.0	0.3
0.8	0.4
1.1	0.5
1.4	0.6
1.6	0.7



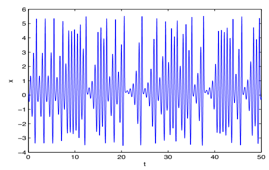
w	t
1.3	0.1
1.2	0.2
1.0	0.3
0.8	0.4
1.1	0.5
1.4	0.6
1.6	0.7



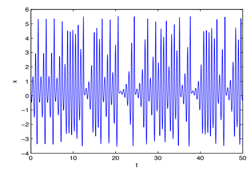
TISEAN’s `delay` command does this

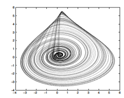


Measure just the *x* coordinate…



…and then embed:

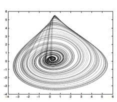




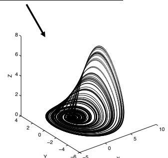
τ = 150

Takens* theorem

For the right τ and enough dimensions, the embedded dynamics are diffeomorphic to (and thus have same topology as) the original state-space dynamics.



τ = 150



* Whitney, Mane, …

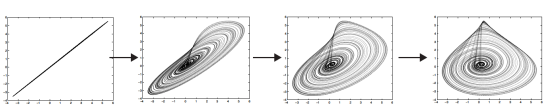
Note: the measured quantity must be a smooth, generic function of at least one state variable, and must be uniformly sampled in time.

Diffeomorphic: mapping from the one to the other is differentiable and has a differentiable inverse.

What that means:

- *qualitatively* the same shape (topology)
- have same dynamical invariants (e.g., λ)

Choosing τ:



τ = 1 → τ = 50 → τ = 100 → τ = 150

TISEAN contains tools that help you do this (e.g., `mutual`)

Choosing *m*

$m > 2d$: sufficient to ensure no crossings in reconstruction space (Takens et al.)…

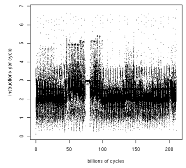
…but that may be overkill, and you rarely know d anyway.

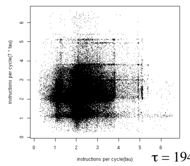
“Embedology” paper: $m > 2\, d_{box}$ (box-counting dimension)

TISEAN contains tools that help you do this (e.g., `false_nearest`)

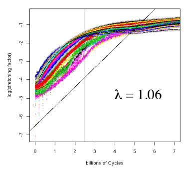
NLTSA* of computer performance dynamics

* nonlinear time-series analysis



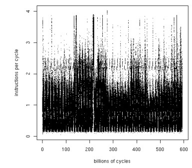


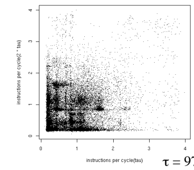
$\tau = 194, m = 10$



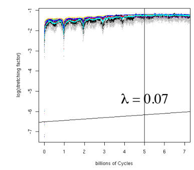
bzip2 dynamics on an Intel Core2

Mytkowicz *et al.*, *Chaos* **19**:033124



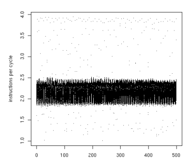


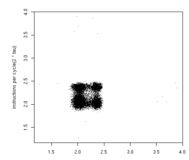
$\tau = 973, m = 12$



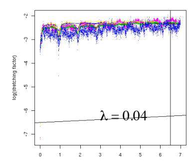
bzip2 dynamics on an Intel Pentium 4

Mytkowicz *et al.*, *Chaos* **19**:033124





$\tau = 930, m = 8$



povray dynamics on an Intel Core2

Mytkowicz *et al.*, *Chaos* **19**:033124

Caveat: need enough data…

If Δt is not uniform

~~Theorem (Takens): for $\tau>0$ and $m > 2d$, reconstructed trajectory is diffeomorphic to the true trajectory~~

~~Conditions: evenly sampled in time, smooth generic measurement function~~

Interspike interval embedding

idea: lots of systems generate spikes — hearts, nerves, etc.

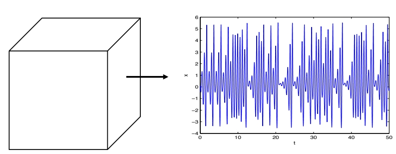
if you assume that the spikes are the result of an integrate-and-fire system, then the Δt has a one-to-one correspondence to some state variable's *integrated* value…

in which case the embedding theorems still hold.

(with the Δ*t*s as state variables)

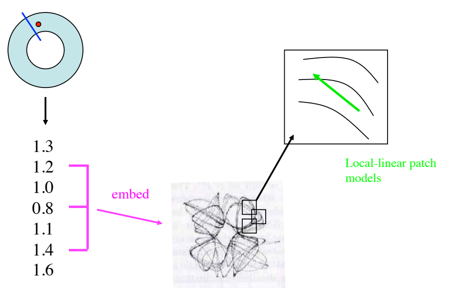
Sauer *Chaos* **5**:127

Prediction



…?

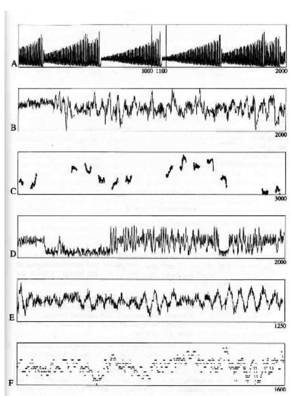
Predicting the path of a roulette ball…



The Eudaemonic Pie
(or The Newtonian Casino)

The Santa Fe competition

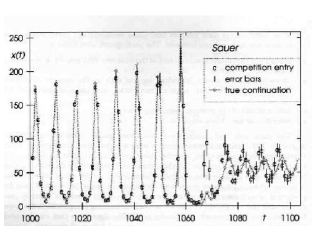
- Weigend & Gershenfeld, 1992
- put a bunch of data sets up on an ftp server
- and invited all comers to predict their future
- chronicled in *Time Series Prediction: Forecasting the Future and Understanding the Past*, Santa Fe Institute, 1993 (from which the images on the following half-dozen slides were reproduced)



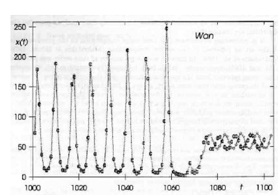
The Santa Fe competition: data

- Laboratory laser
- Medical data (sleep apnea)
- Currency rate exchange
- RK4 on some chaotic ODE
- Intensity of some star
- A Bach fugue

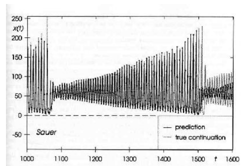
Embedding + patch models: (Sauer)



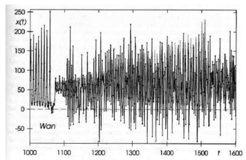
Neural net: (Wan)



Further out:

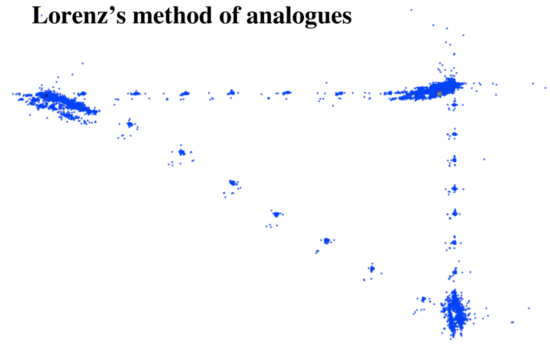


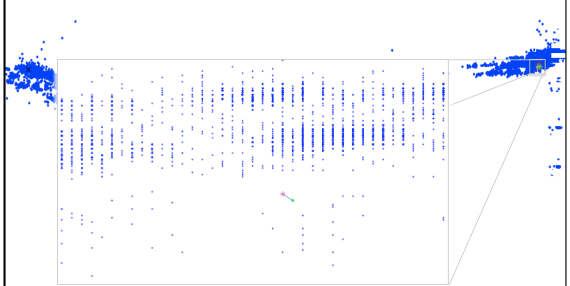
Sauer



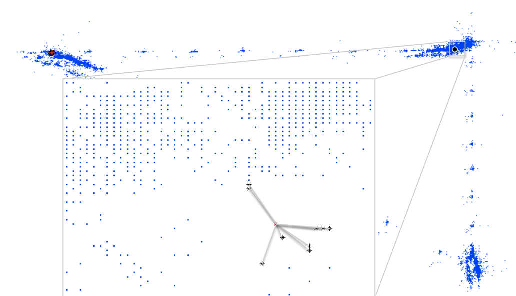
Wan

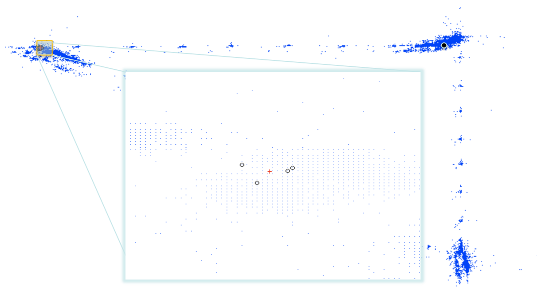
**An even simpler prediction method:
Lorenz's method of analogues**

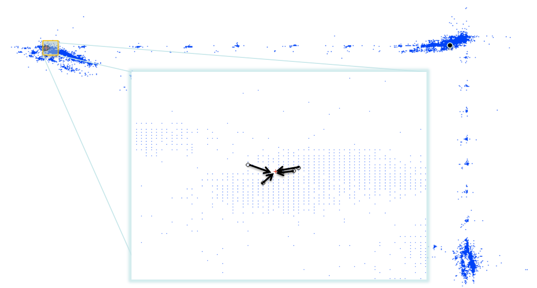




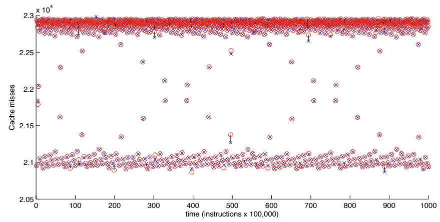
A *k*-nearest neighbor modification of LMA







Using *k*LMA to predict computer dynamics



RMSPE = 11.566
(.07% of the average)

`column_major`
cache misses

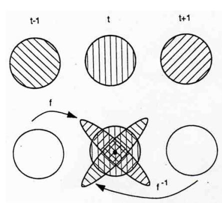
Garland/Bradley *Intelligent Data Analysis* 2011

Noise…

Linear filtering: a bad idea if the system is chaotic

Nonlinear alternatives:

- use the stable and unstable manifold structure on a chaotic attractor…

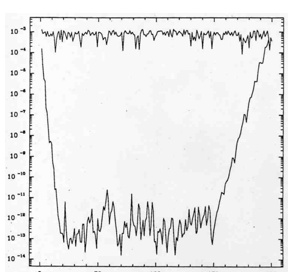


Farmer & Sidorowich, in *Evolution, Learning and Cognition*, World Scientific, 1983

Idea:

- If you have a model of the system, you can simulate what happens to each point in forward *and backward* time
- If your system has transverse stable and unstable manifolds, that does useful things to the noise balls
- Since all three versions of that data should be identical at the middle time, can average them
- ⇨ noise reduction!
- Works best if manifolds are perpendicular, but requires only transversality

Results:



Farmer & Sidorowich, in *Evolution, Learning and Cognition*, World Scientific, 1983

Noise…

Linear filtering: a bad idea if the system is chaotic

Nonlinear alternatives:

- use the stable and unstable manifold geometry on a chaotic attractor
- what about using the *topology* of the attractor?

Computational Topology

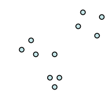
Why: this is the fundamental mathematics of shape. complements geometry.

What: compute topological properties *from finite data*

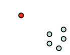
How:
- introduce resolution parameter
- count components and holes at different resolutions
- deduce topology from patterns therein

V. Robins Ph.D. thesis, UColorado, 1999

Connectedness: definitions

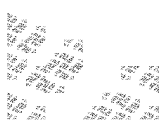


- how many "lumps" in a data set:
- ε-connectedness (after Cantor)
- ε-connected components
- ε-isolated points:

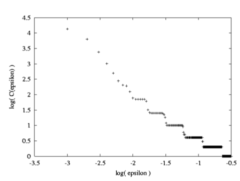


Connectedness: examples

If the data points are samples of a disconnected fractal like this:



The number of connected components looks like this:

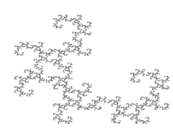


(note obvious tie-in to fractal dimension…)

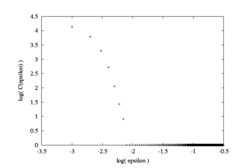
Robins *et al.*, *Physica D* **139**:276, *Nonlinearity* **11**:913

Connectedness: examples

If the data points are samples of a connected set like this:



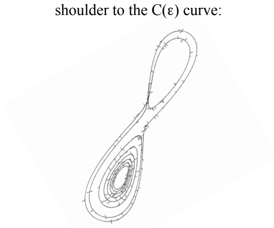
The number of connected components looks like this:

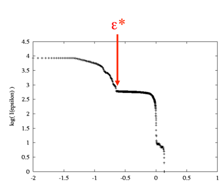


Robins *et al.*, *Physica D* **139**:276, *Nonlinearity* **11**:913

Connectedness and filtering

The effect of noise is to add isolated points to the set and a shoulder to the C(ε) curve:

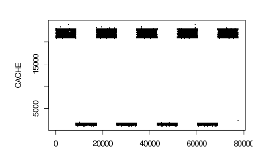


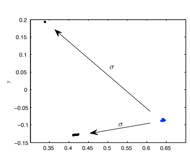


So if you know that the object is connected — like the attractor of a flow — you can reasonably assume that any isolated points are noisy, and remove them by pruning with ε = ε*

Robins *et al.*, *Intelligent Data Analysis* **8**:505, *Chaos* **14**:305

Continuity and filtering





Idea:
- deterministic, differentiable dynamics (maps & flows) are *continuous*

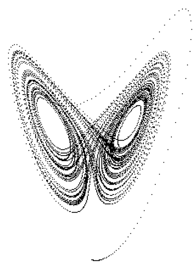
Conjecture:
- if the image of a connected set is not connected, more than one dynamics is at work

Approach:
- track connectedness over time

Applications:
- pulling apart interleaved dynamics, removing noise…

Alexander *et al.*, *CHAOS.*, 2012

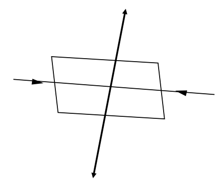
Chaos and control



key concepts:

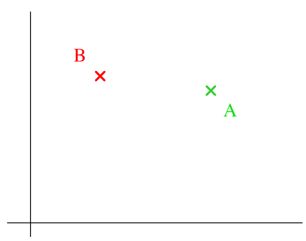
- dense attractor coverage
- exponential trajectory separation
- un/stable manifold structure
- local-linear control

Recall: local-linear control of a saddle point works successfully in a region whose geometry is defined by the λ_i and the W^s/W^u, together with the sensor & actuator capabilities…



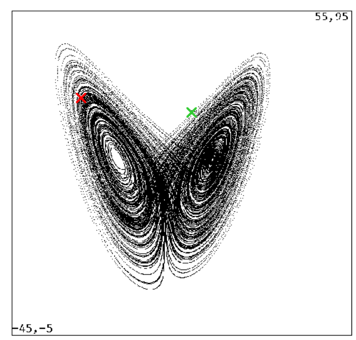
Control:

getting from A to B, minimizing some cost functional



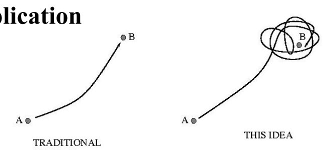
Lorenz System:

denseness, reachability, and control



R = 50

Denseness & reachability in a real engineering application

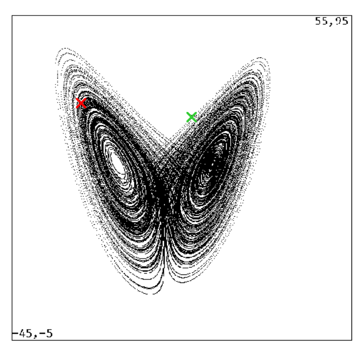


- can control position/volume/density of attractor — *within limits*
- possibly not reachable any other way
- not for time-critical applications (that "eventually")

Using Chaos to Broaden the Capture Range of a Phase-Locked Loop

Elizabeth Bradley, *Member, IEEE*

Now what?



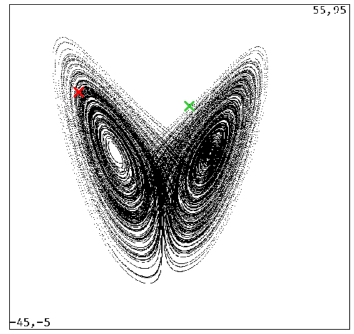
R = 50

OGY control

- dense attractor coverage → reachability
- un/stable manifold structure + UPO denseness + local-linear control → controllability

Ott *et al.*, *PRL* **64**:1196

Use local-linear control, designed using the eigen-values and eigen-vectors at that point × to balance a chaotic system on a UPO passing through that point.



But you're relying on denseness to get you into the controllable region, and that may take a while…

- dense attractor coverage → reachability
- un/stable manifold structure + UPO denseness + local-linear control → controllability
- exploit sensitive dependence, too???

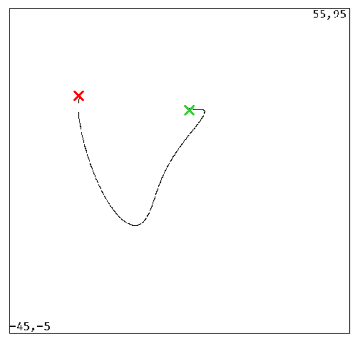
⟹ "targeting"

Lorenz System:

SDOIC-based targeting

OGY & co. have been used in *tons* of systems; see Shinbrot review paper.

Alfred Hubler has done a lot of cool stuff in this area as well.



Four R switches; 240X faster

Bradley, *Cybernetics & Systems* **26**:299

Program in Applied Mathematics

Erik Bollt

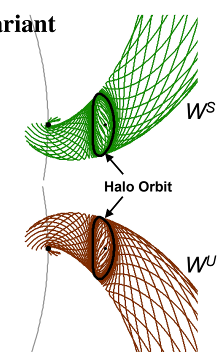
University of Colorado at Boulder
Boulder CO 80309-0526
(303) 492-4668

Other cool ways to use invariant manifolds

Want to get a spacecraft onto a "halo orbit," which is a UPO of the dynamics.

Unstable Periodic Orbits (UPOs) have invariant manifolds:

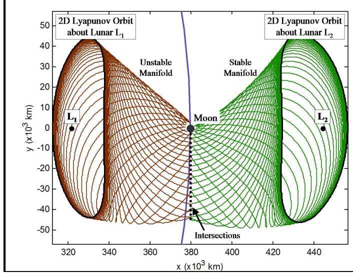
- Stable Invariant Manifold (W^S)
  - *The set of all trajectories a particle could use to arrive onto the UPO.*
- Unstable Invariant Manifold (W^U)
  - *The set of all trajectories a particle could take after a small perturbation from the UPO.*

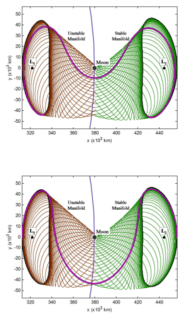


Jeff Parker, PhD thesis, UColorado 2008

Low-energy (cheap) orbit transfers

- Depart along W^U_{L1} & arrive on W^S_{L2}

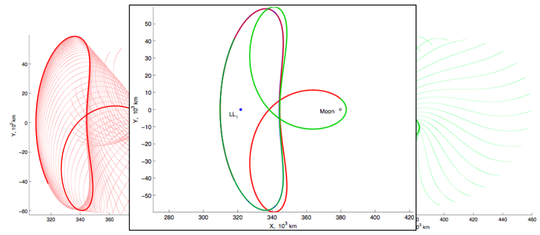




Jeff Parker, PhD thesis, UColorado 2008

Homoclinic orbits - The best case

- If a trajectory in Stable and Unstable intersect ("homoclinic connection")

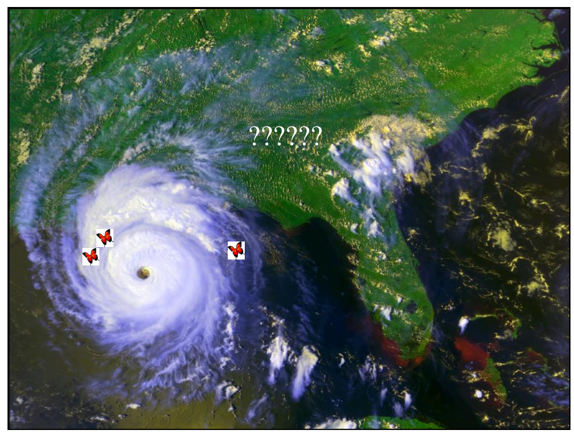


Unstable Manifold of an LL_1 Lyapunov Orbit   Stable Manifold of an LL_1 Lyapunov Orbit

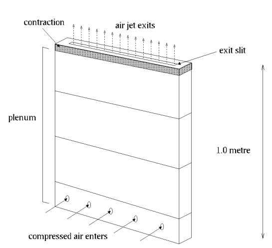
Jeff Parker, PhD thesis, UColorado 2008

<u>Can we do any of that in spatially extended systems?</u>

(i.e. harness the butterfly effect, exploit un/stable manifold geometry?)

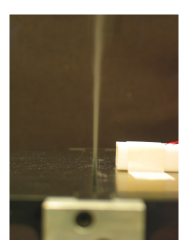


A 2D jet

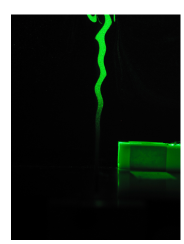


Peacock *et al.*, *Exp. Fluids* **37**:22

End view



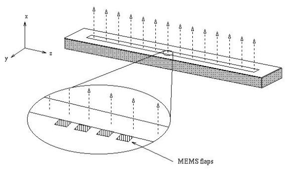
room lighting



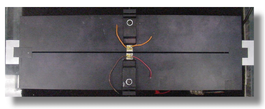
stop-action laser "slice"

aerosolized canola oil

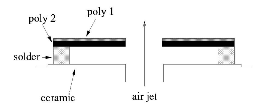
Forcing the jet flow

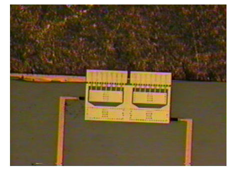


Slit: 2.5 X 400 mm



MEMS actuators





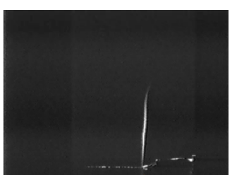
Video : overhead view at 2Hz, 10V

Area of individual flap is 1.0 x 0.25mm

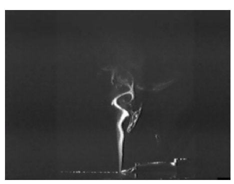
Ma *et al.*, *IEEE Trans. Adv. Packaging* **26**:268

The Butterfly effect in action…

no forcing



6Hz forcing

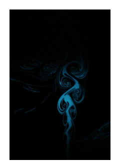


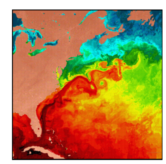
MEMS flap

Forcing generates coherent structures that enhance entrainment and mixing

Peacock *et al.*, *Exp. Fluids* **37**:22

Does this have anything to do with reality?





Measurement & isolation:

Communication and chaos:

- Two coupled Lorenz systems will synchronize
- Robust to a small amount of noise
- Use this to transmit & receive information

Transmitter		Receiver
$x' = a(y-x)$		$x' = a(y-x)$
$y' = rx - y - xz$	→	$y' = r(x+\varepsilon x) - y - xz$
$z' = xy - bz$		$z' = xy - bz$

- Chaotic carrier wave, so hard to intercept or jam

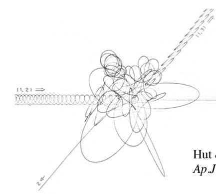
Pecora & Carroll *Phys. Rev. Lett* **64**:821

Another interesting application: chaos in the solar system

- orbits of Pluto, Mars
- Kirkwood gaps
- rotation of Hyperion & other satellites
- …

Solar system stability:

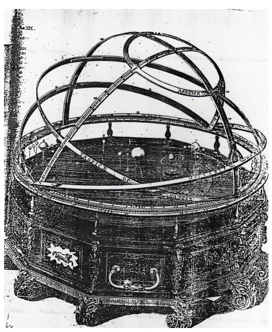
- recall: two-body problem not chaotic
- but three (or more) can be…



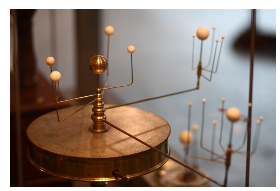
Hut & Bahcall
Ap.J. **268**:*319*

Exploring that issue, circa 1880:

An *orrery*, which is a *mechanical computer* whose gear ratios and circular platters simulate the orbits of the planets



Exploring that issue before the digital computer age…

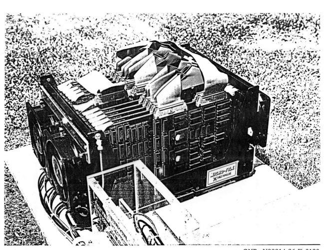


An *orrery*, which is a *mechanical computer* whose gear ratios and circular platters simulate the orbits of the planets

Exploring that issue, circa 1980:

- write the *n*-body equations for the solar system
- solve them using symplectic ODE solvers on a special-purpose computer

The *digital orrery*
(Wisdom & Sussman)



ONR N00014-86-K-0180

Numerical Evidence That the Motion of Pluto Is Chaotic

GERALD JAY SUSSMAN AND JACK WISDOM

The Digital Orrery has been used to perform an integration of the motion of the outer planets for 845 million years. This integration indicates that the long-term motion of the planet Pluto is chaotic. Nearby trajectories diverge exponentially with an *e*-folding time of only about 20 million years.

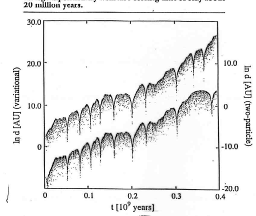


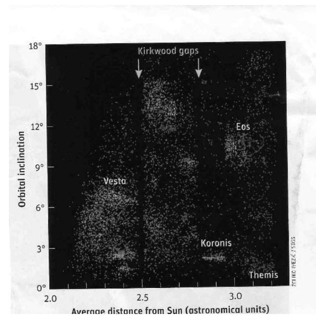
Figure 3: The exponential divergence of nearby trajectories is indicated by the average linear growth of the logarithms of the distance measures as a function of time. In the upper trace we see the growth of the variational distance around a reference trajectory. In the lower trace we see how two Plutos diverge with time. The distance saturates near 80AU when the Plutos are on opposite sides of the Sun. The variational method of studying neighboring trajectories does not have the problem of saturation. Note that the two methods are in excellent agreement until the two-trajectory method has nearly saturated.

Science **241**:433

Should we worry?

- No.

Kirkwood gaps:



From *Sky & Telescope*

Chaos and the Kirkwood gaps

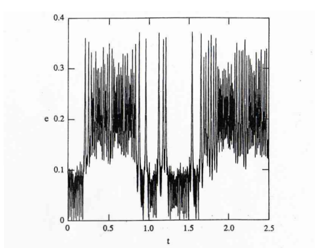


FIGURE 5. Eccentricity of a typical chaotic trajectory over a longer time interval. the time is now measured in millions of years. Bursts of high eccentricity behavior are interspersed with intervals of irregular low eccentricity behavior, broken by occasional spikes.

Wisdom, *Nuclear Phys. B* **2**:391

Evidence in favor of the conjecture:

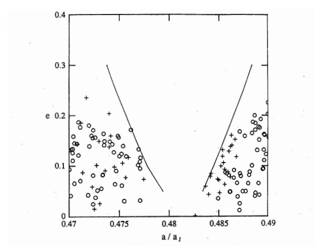


FIGURE 9. Comparison of the actual distribution of asteroids with the outer boundaries of the chaotic zone. There is both a chaotic region and quasiperiodic region in the gap, but trajectories of both types are planet crossing.

Wisdom, *Nuclear Phys. B* **2**:391

Chaotic tumbling of satellites:

Voyager and Galileo saw this…

From *Sky & Telescope*

Ap. J. **97**:570
Ap. J. **98**:1855

…so did Cassini:

www.nasa.gov/mission_pages/cassini/multimedia/pia06243.html

Chaotic tumbling of satellites:

This happens for **all** satellites at some point in their history, unless they are perfectly spherical and in perfectly circular orbits (pf: KAM theorem; see Wisdom paper on syllabus)

Some of them are still tumbling chaotically because of their geometry, but most (like the earth and its moon) have settled down into tidal equilibria

More chaos in the solar system:

- obliquity of Mars (Touma & Wisdom, *Science* **259**:1294)

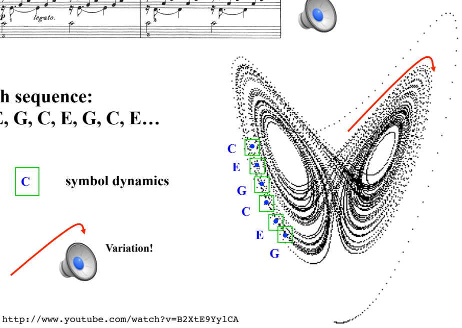
www.solarviews.com

- etc.

Musical Variations from a Chaotic Mapping

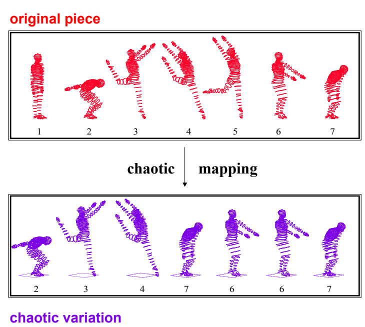
Dabby *Chaos* **6**:95

Pitch sequence:
C, E, G, C, E, G, C, E…

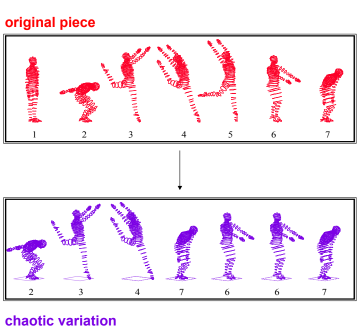


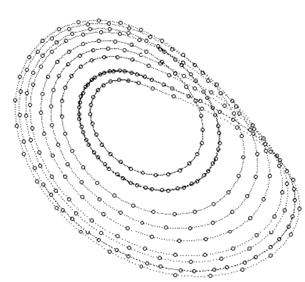
Also fun: http://www.youtube.com/watch?v=B2XtE9Yy1CA

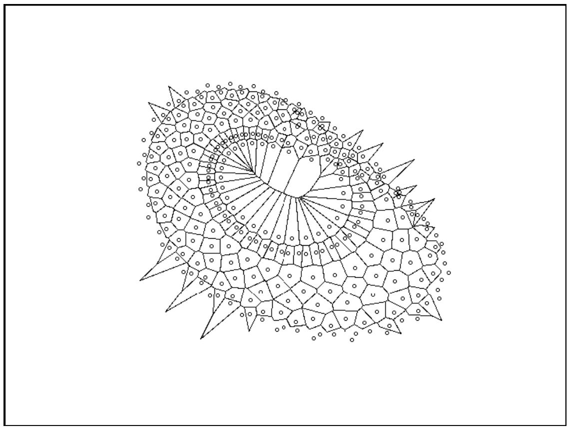
Chaotic variations on movement sequences

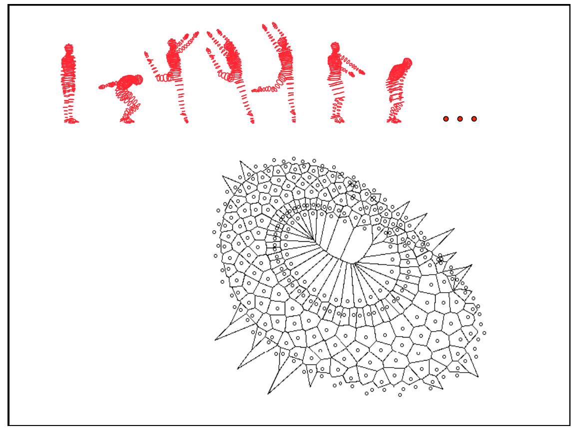


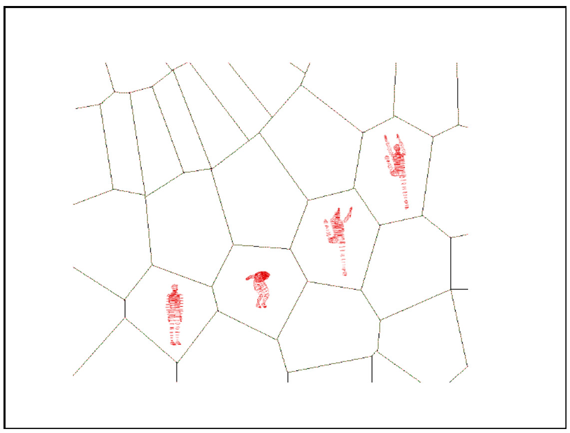
Bradley & Stuart, *Chaos* **8**:800

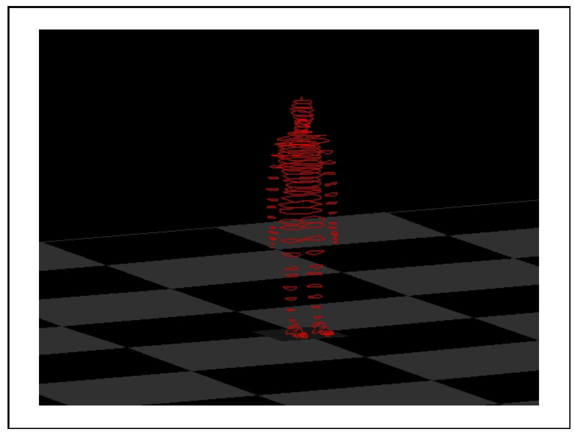


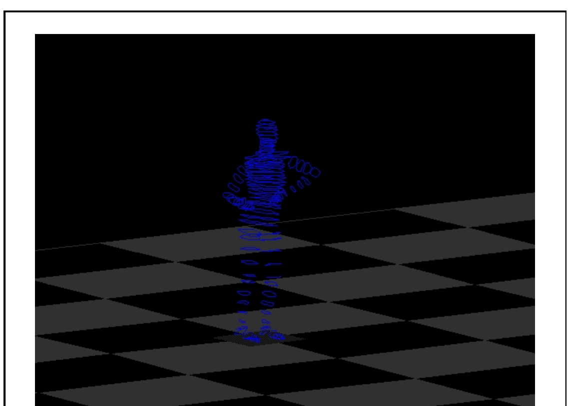




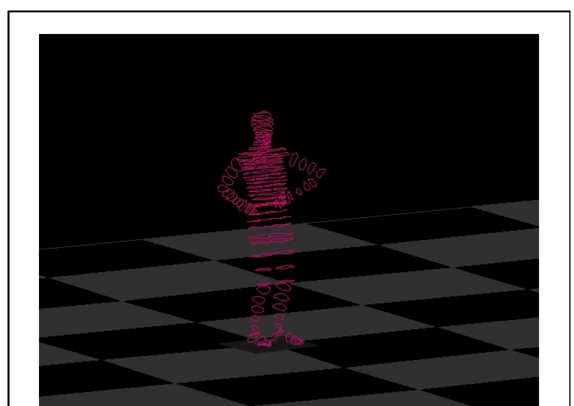




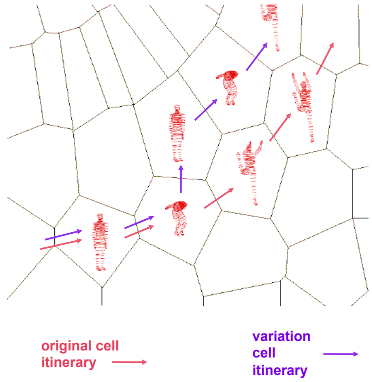


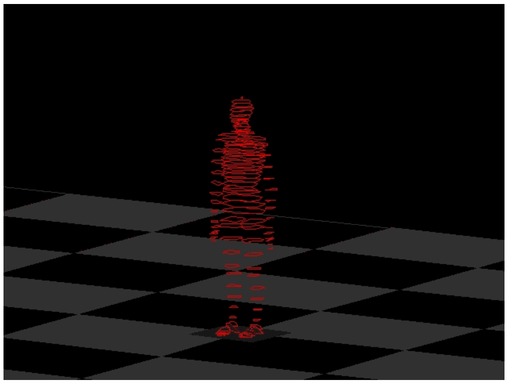


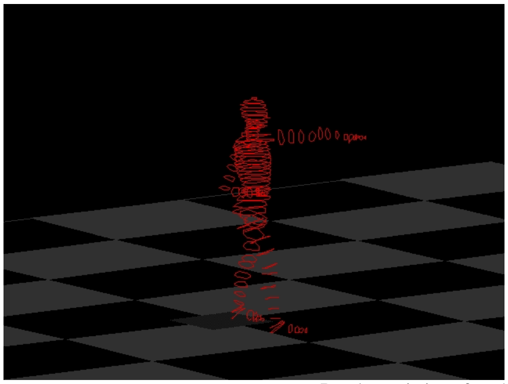
Lorenz

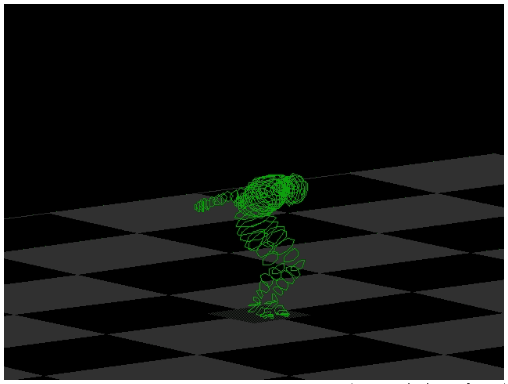


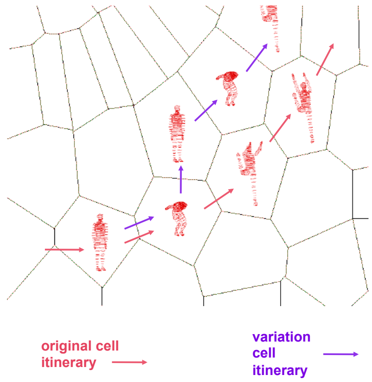
Rossler

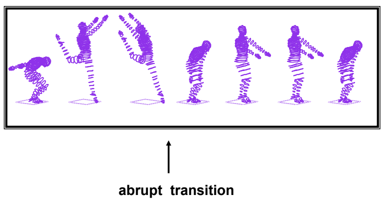


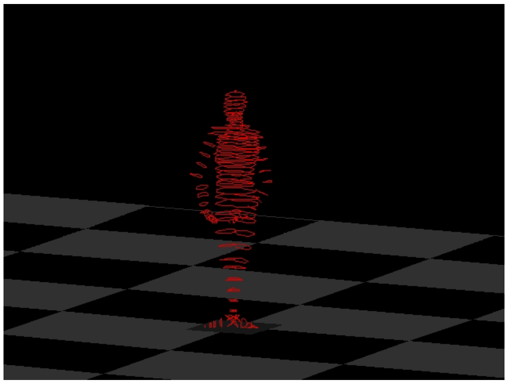

medley—original

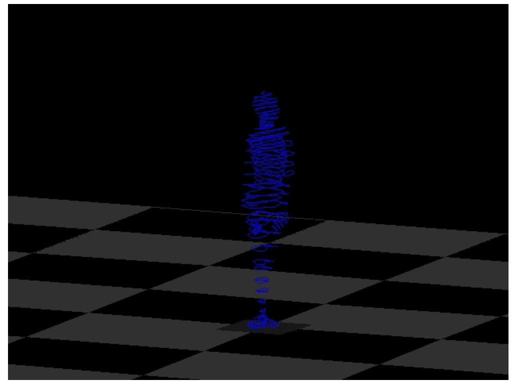

Rossler variation of medley


random variation of medley

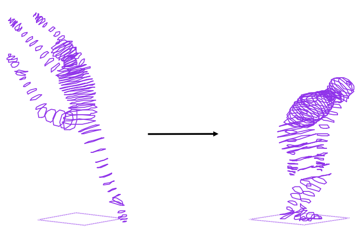






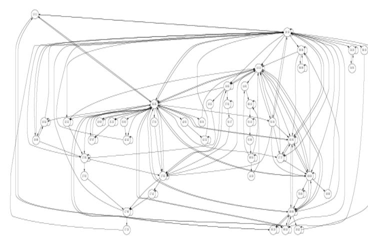


Interpolation



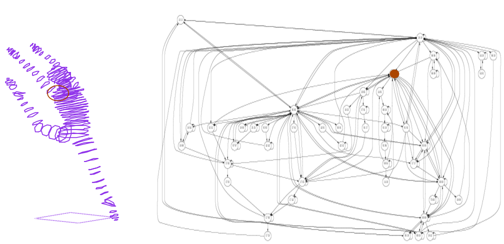
Stuart & Bradley, *ICML* 1998

Corpus-based approach

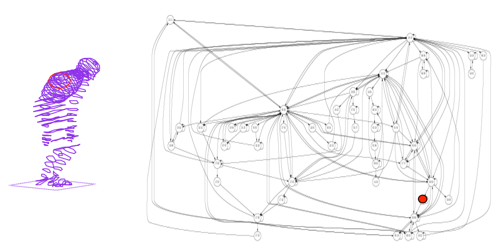


- graph captures motions of one joint
- note: specific to the genre of the corpus!

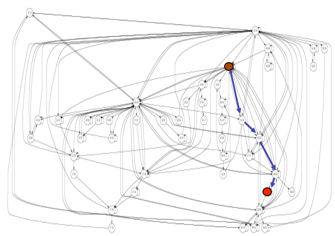
Initial state



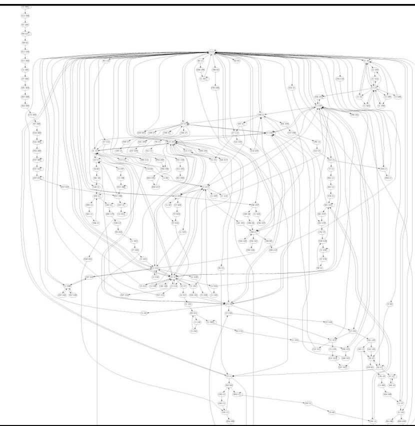
Target state

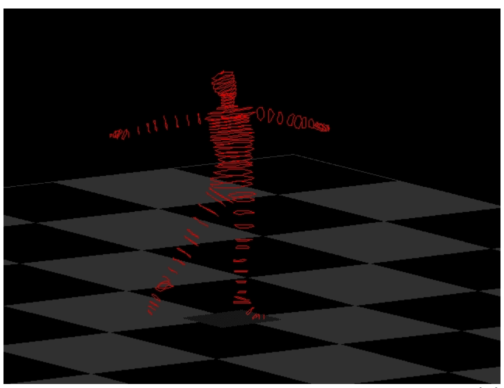


Graph search



…for 44 joints in parallel!





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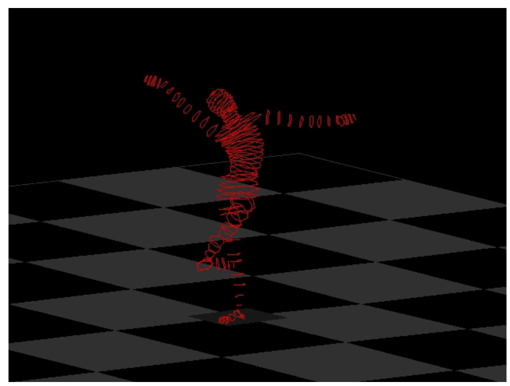
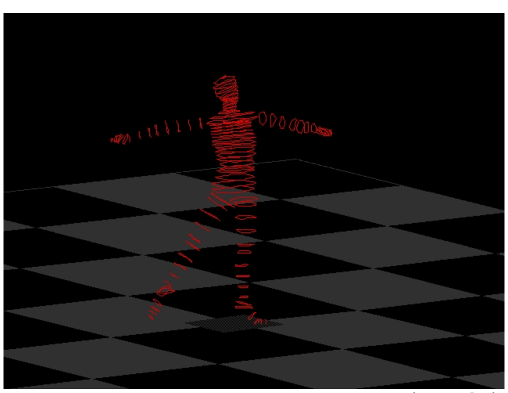
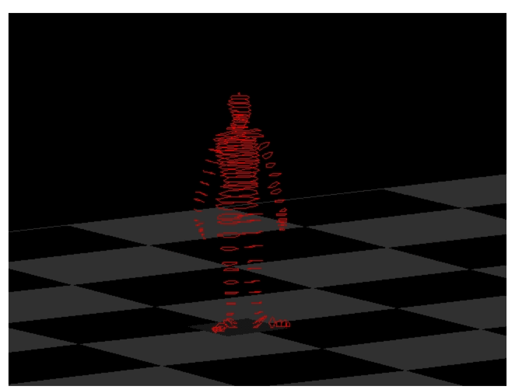


target



interpolation 1



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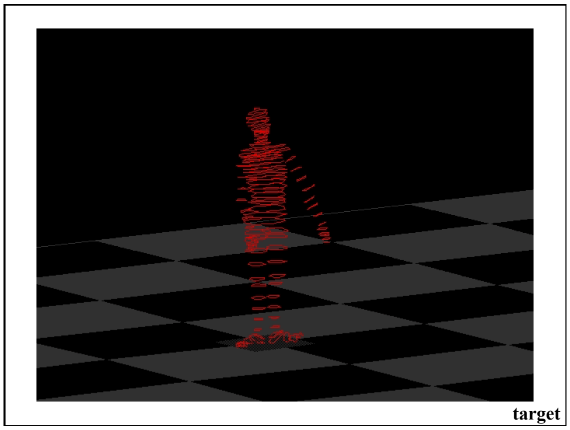
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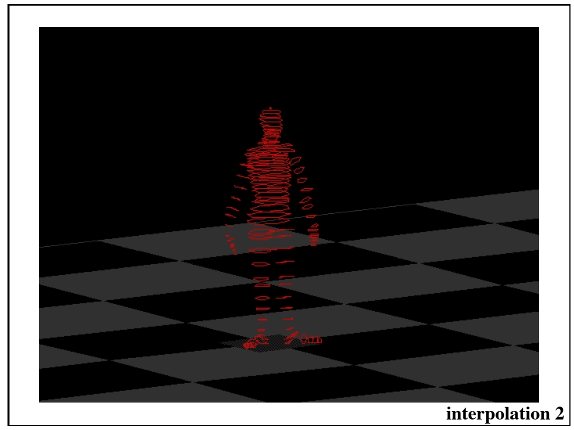
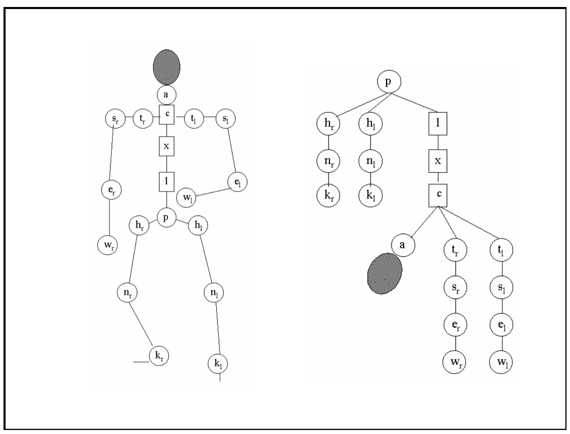
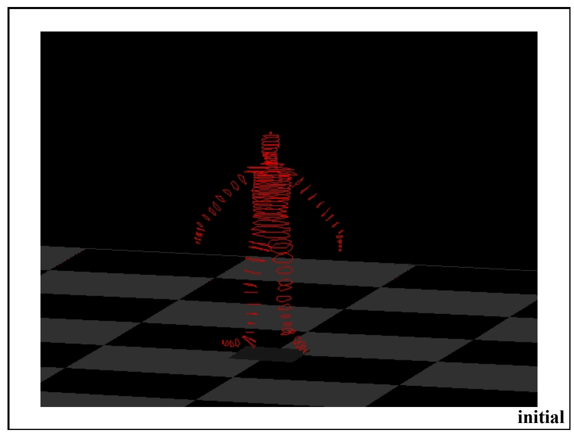

interpolation 2

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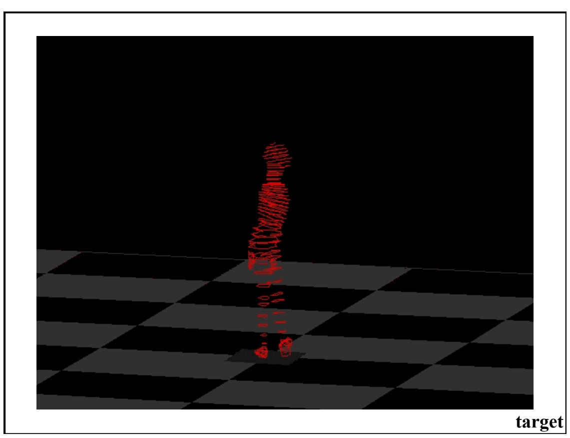
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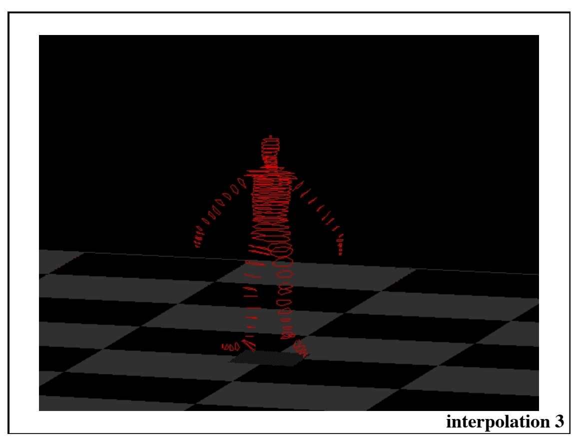
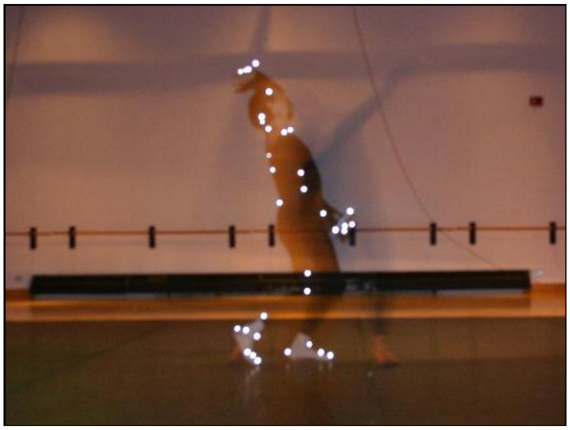
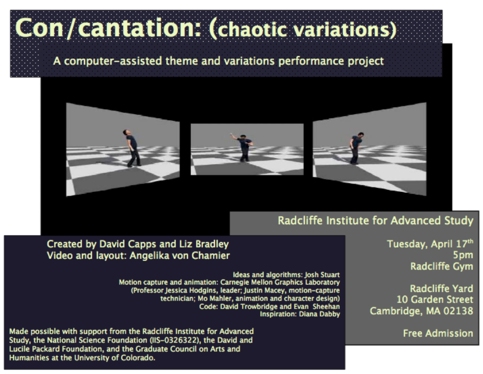
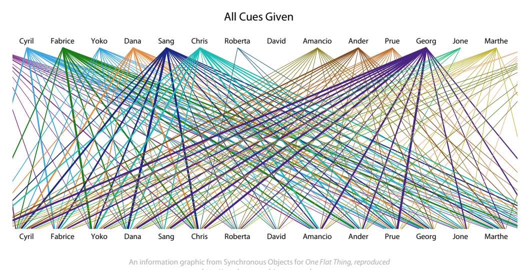

interpolation 3







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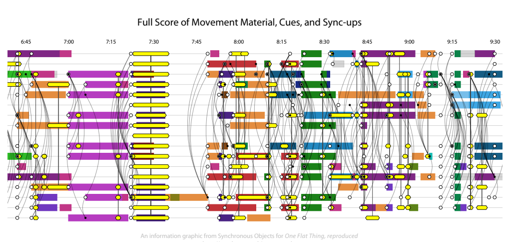

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