Bayesian Reasoning for Intelligent People

Simon DeDeo*

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1 The Bayesian Angel

"Bayesian reasoning" is a fancy phrase for "the use of probabilities to represent degrees of belief, and the manipulation of those probabilities in accordance with the standard rules." You learned many of the standard rules for manipulating probability in high school; you can find a derivation of them in Ref. [1] (Lecture Four, "Laplace's Model of Common Sense"). There are, in fact, many ways to derive them, ranging from the philosophical ("consistent reason requires us to do this"), to the evolutionary ("populations whose members approximate these rules better, grow more quickly"),

*Social and Decision Sciences, Carnegie Mellon University & the Santa Fe Institute. Current version available at `http://santafe.edu/~simon/br.pdf`. Please send corrections, comments and feedback to `simon@santafe.edu`; `http://santafe.edu/~simon`.

to the economic ("if you don't do this, I can induce you to place a series of bets with me that you are guaranteed to lose.")

The most basic of these rules is how to turn joint probabilities into conditional probabilities. We have the following identity:

$$P(x, y) = P(x|y)P(y) = P(y|x)P(x), \tag{1}$$

where $P(x, y)$ is notation for "the probability of both x and y being true" (the "joint" probability), $P(x|y)$ is notation for "the probability of x being true *conditional* on (or given that) y being true", and $P(x)$ is notation for "the probability that x is true". Here x and y are shorthand for sentences; for example, x might stand for "the student chosen at random is blond", and y might stand for "the student chosen at random is male." You should try saying Eq. 1 out loud, using explicit sentences for x and y, to see if it sounds reasonable to you.

That's it. Just thinking carefully about how to set up problems, and using this equation at critical moments, will make you an ideal reasoner—in an important sense, an optimal reasoner.[1] Using Bayes' theorem might make you an optimal reasoner, but it's not something an ordinary person can do all the time, at least, not perfectly. It might, for example, require keeping track of gigantic lists of conditional probabilities and manipulating them at will, and that's not something we evolved well to do. Sometimes, to emphasize the idealized nature of Bayesian descriptions of reasoning, and to contrast them with the messier kind of approximations (or forgeries) that human beings do, we'll talk about a *Bayesian angel* with infinite memory and processing power (but, importantly, not infinite perceptual abilities nor experience—like bodhisattvas, Bayesian angels live in the real world).

While Bayesian reasoning makes no changes to how you follow the rules you learned in high school, it does ask you to make a fundamental shift in how you think about them. You are likely used to thinking about probabilities in terms of frequencies: if the probability of an event x is 0.5, you expect it to happen "about 50% of the time". We call this the *frequentist* perspective, though it comes so naturally to us we can forget it's a deliberate choice. Often times the frequentist perspective is good enough: if you have a coin, you can toss it multiple times to see what happens and use the number of times it comes up heads to attribute an probability to the coin itself. Frequencies are facts about the world, and so the frequentist can be said to think of probabilities as themselves "objective" facts about the world.

Bayesians flip the problem around: instead of seeing probabilities as out there in the world, a property of objects like coins, they understand them as describing subjective states of belief that an observer might have. If an individual attributes a probability p to event x, this is now understood as indicating that the observer has "degree of belief p" in the event's taking place (or, indeed, in having taken place). If p is close to one, the individual is very sure x is true (or will happen); if p is close to zero, the individual is very sure it's not the case. If p is precisely zero, the individual considers it absolutely impossible; as we shall see, this means that (for example) no evidence of any form whatsoever will cause them to raise p above zero.

The shift from "frequentist" to "subjective" probability is not an easy one to make. But it's absolutely essential to do it if you want to understand the Bayesian revolution in cognitive science, where it plays a core role in modeling actual states of belief in real-world agents—or, indeed, if you want the ways in which these ideas have revolutionized artificial intelligence, machine learning, and data analysis in the modern era.

[1] You can be an optimal reasoner with the wrong facts, or an optimal reasoner with the right facts and the wrong belief space (set of sentences). We'll see examples of both later in this guide. Being optimal doesn't mean being *right* about anything.

The fact that Bayesians understand probabilities as describing a subjective, reasoning process may seem to undermine the "objectivity" of the statements it makes. In fact, it turns out to be a huge advantage: think, for example, of predicting the outcome of an election. If we're told our candidate has a 70% chance of winning, we want to know why—on what basis—was that prediction made? We want to see how the degree of belief the statistician urges on us is a combination of other things we might believe, given polling data, say, and theories, which we may only have limited confidence in, about how to interpret it. Bayesian tools lift the cover on this process, laying the machinery of thought bare for inspection.

2 Bayes' Theorem and Madame Blavatsky

The identity Eq. 1 is the basis of something famously known as "Bayes' Theorem". You just divide both sides by $P(y)$:

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)} \tag{2}$$

Bayes' Theorem is something you can use to win arguments against statistics nerds, and telling them they've violated it is sort of like telling an ordinary person that they have a rip in the seat of their pants. They are going to want to check that out right away. Bayes' Theorem gets its power from how it can invert a kind of question you tend to ask, but can't answer, into a question that sounds a bit weird, but it turns out you can answer. We'll use T and D as our random variables now, where T stands for a set of (exhaustive and mutually exclusive) theories about the world, and D for the kind of data you might get.

We often want to know what our degree of belief should be in a theory t, given some data d. This is a profound problem when the theories are of great import and the data expensive and hard to gather. To have some fun, we'll do some toy problems: made-up scenarios that will help you build your intuition and give you the skills to reason when it matters. We begin with an example drawn from the physicist E.T. Jaynes, in the "Queer Uses of Probability" chapter of Ref. [1].

Our friend Artemy[2] returns from a trip to New York City, reporting that he saw Madame Blavatsky, the famous clairvoyant, successfully predict the outcome of 100 coin tosses. Should we believe in ESP, the theory that some people have a magical ability to sense the future?

We start by setting up our T and D. T is {"ESP is real", "ESP is not real"}. D is {"Madame Blavatsky is no better than chance at predicting the toss of an unbiased coin", "Madame Blavatsky can predict perfectly the outcome of 100 coin tosses"} (for simplicity, we'll assume these are the only possible kinds of data we can get—another way to put it is that we'll attribute probability zero to any other kind of data). We'll abbreviate these as T={ESP,∼ESP} and D={normal, predict}; the little ∼ is shorthand for "not". We want to know the following: given that Madame Blavatsky did this amazing thing, what should I believe about ESP? More formally, "conditional on *predict*, what degree of belief should I have in *ESP*?" Using a rearrangement of Eq. 1, we have

$$P(\text{ESP}|\text{predict}) = \frac{P(\text{predict}|\text{ESP})P(\text{ESP})}{P(\text{predict})}. \tag{3}$$

We'll go term by term on the right-hand side. $P(\text{predict}|\text{ESP})$ means "what's the chance that we get the data {Madame Blavatsky predicts perfectly} given the truth of the theory ESP." Let's say that if ESP is real, Madame Blavatsky almost certainly has it, and if she has it, she can do amazing

[2] A parallel universe version of one-time fearless Associate Instructor at IU, now postdoctoral fellow at the Santa Fe Institute, Artemy Kolchinsky.

predictions like these, so we set that at 0.9—*i.e.*, only a 10% chance she'll screw up using her (real) magic powers.

P(ESP) is the prior belief you have in ESP—the degree of belief you attribute to the possibility before hearing about the new data. Let's say you're a scientist; you attribute low value to these kinds of things, but (you're a scientist)—nothing is impossible, so we'll say 10^{-12}. You're more confident that ESP is fake than you are about surviving your next airline flight.[3]

Finally, P(predict): the probability this prediction event happens. I always find this term hard to think about, but then I just recall that $P(\text{ESP}|\text{predict}) + P(\sim \text{ESP}|\text{predict})$ has to sum to unity.[4]

$$\begin{aligned} P(\text{predict}) &= P(\text{predict}|\text{ESP})P(\text{ESP}) \\ &\quad + P(\text{predict}|\sim \text{ESP})P(\sim \text{ESP}). \end{aligned}$$

$P(\sim \text{ESP})$ is easy—that's just $1 - P(\text{ESP})$, or $1 - 10^{-12}$. $P(\text{predict}|\sim \text{ESP})$ is the chance of guessing one hundred coin tosses in a row, given the fact that it's impossible to see the future (so you have to guess). That is just $0.5 \times 0.5 \times \ldots$—you have a fifty-fifty chance the first time, times a fifty-fifty chance the second, and so forth. For 100 tosses, it's 2^{-100}, or about 7×10^{-31}. We're set—we can now plug in all the numbers to discover that

$$P(\text{ESP}|\text{predict}) = \frac{0.9 \times 10^{-12}}{0.9 \times 10^{-12} + 7 \times 10^{-31}(1 - 10^{-12})} \approx 1 - 10^{-18}, \tag{4}$$

or, in words: ESP is almost certainly true (very very very close to one), conditional on Madame Blavatsky's performance.

3 Observer Reliability and Hume's Argument against Miracles

Or is it? Let's go back to our theory set, T, and enlarge it. What if we allowed for another possibility: our friend Artemy is delusional, or was gulled by a stage magician? T is now {"ESP is real, Artemy is not crazy", "ESP is not real, Artemy is not crazy', "ESP is real, Artemy is crazy", "ESP is not real, Artemy is crazy"}. We'll abbreviate as before: {ESP&ANC, ∼ESP&ANC, ESP&AC, ∼ESP&AC}. We can assume that our theories have two independent parts, so that we can write

$$P(\text{ESP\&ANC}) = P(\text{ESP})P(\text{ANC}). \tag{5}$$

If the prior didn't decompose, it would mean that somehow these two features of the world would be connected. In an ESP world, for example, Artemy might be more likely to be crazy.

[3]You might say "it's more likely I'll die in a plane crash on my next flight than ESP turns out to be real." True Bayesians would object to this statement, because I'm implicitly referring to a frequentist notion—"'what *would* happen *if* you tried something"; even worse, implicitly, "how often would it happen if you tried it lots of times". Frequentists are the ancient and heretical pagans of probability, whose practices were superseded by the unitary Bayesian religion. To be admitted to the Church of Bayes, you must remain entirely in the world of degrees of belief. Note that in this case, I (Simon) *used* counts of airline crashes to derive a degree of belief—but that's OK, and I'm still a Bayesian, because secretly I did something a bit meta: I used the counts to test different hypotheses about the degree of belief I should have in airline safety, and chose the degree of belief about airline safety that I had the highest degree of belief in! I used a Dirichlet prior with $\alpha = 1$ (now do you believe me). Bayesians are allowed to have data commerce with Frequentists. They just can't share beliefs—because, a True Bayesian would say, Frequentists are fundamentally mistaken about the nature of beliefs they have, and so can't communicate them to you. Frequentists will, for example, eventually give contradictory answers to logically identical questions.

[4]Rewrite Eq. 3 for $P(\sim \text{ESP}|\text{predict})$, set the sum of that and Eq. 3 equal to 1, and solve for P(predict).

We consider Artemy a usually very reliable guy, so the chance that he's crazy, let's say, is 10^{-6}. It's very unlikely that Artemy is crazy—about ten times less than the lifetime chance of dying from being struck by lightning.[5] Now,

$$P(\text{ESP\&ANC}|\text{predict}) = \frac{P(\text{predict}|\text{ESP\&ANC})P(\text{ESP})P(\text{ANC})}{P(\text{predict})}. \tag{6}$$

Let's say that if Artemy is not crazy, but ESP is real, $P(\text{predict}|\text{ESP\&ANC})$ is just $P(\text{predict}|\text{ESP})$—we previously agreed that that was 0.9. But now the denominator has changed—we'll compute $P(\text{predict})$ by summing over four possibilities, not two. Rather than be tedious about it, let's ask a slightly different question. What's the *odds-ratio* of ESP&ANC vs. ∼ESP&AC? How much more likely is it that ESP is false, and Artemy is crazy, rather than ESP is true and Artemy not crazy? We'll just divide the two,

$$\frac{P(\sim \text{ESP\&AC}|\text{predict})}{P(\text{ESP\&ANC}|\text{predict})} = \frac{P(\text{predict}|\sim \text{ESP\&AC})P(\sim \text{ESP})P(\text{AC})}{P(\text{predict}|\text{ESP\&ANC})P(\text{ESP})P(\text{ANC})}, \tag{7}$$

where (to be clear) we know the denominator from Eq. 6, we had to use Eq. 1 to get the numerator, and that annoying $P(\text{predict})$ cancelled. We just need to specify our theory of what happens when ESP is false, but Artemy is crazy. In this case, let's say, Artemy can be led to believe crazy things by a pseudomagician without much difficultly, and $P(\text{predict}|\sim \text{ESP\&AC})$ is, let's say, 0.9.

$$\frac{P(\sim \text{ESP\&AC}|\text{predict})}{P(\text{ESP\&ANC}|\text{predict})} = \frac{0.9 \times (1 - 10^{-12}) \times 10^{-6}}{0.9 \times 10^{-12} \times (1 - 10^{-6})} \approx 10^{6}, \tag{8}$$

or, in words: it's a million times more likely that Artemy is crazy, than ESP is real.

This mathematical result was anticipated well before Bayes' rule gained the fame is has today. The famous free-thinker, philosopher, and Scotsman, David Hume wrote in his essay "Of Miracles" in 1784,

> The plain consequence is (and it is a general maxim worthy of our attention), "That no testimony is sufficient to establish a miracle, unless the testimony be of such a kind, that its falsehood would be more miraculous, than the fact, which it endeavours to establish: And even in that case, there is a mutual destruction of arguments, and the superior only gives us an assurance suitable to that degree of force, which remains, after deducting the inferior."
>
> When anyone tells me, that he saw a dead man restored to life, I immediately consider with myself, whether it be more probable, that this person should either deceive or be deceived, or that the fact, which he relates, should really have happened. I weigh the one miracle against the other; and according to the superiority, which I discover, I pronounce my decision, and always reject the greater miracle. If the falsehood of his testimony would be more miraculous, than the event which he relates; then, and not till then, can he pretend to command my belief or opinion.

Notice one last, sad, thing about Artemy and ESP. Say Artemy tells me about the ESP he saw, and my priors mean that I think he's crazy. He gets upset—and comes back the next day. "You didn't believe me yesterday, Simon, but I went back and *I saw her do it all over again.* Now will

[5]National Safety Council, *Injury Facts 2013.* `http://www.nsc.org/news_resources/injury_and_death_statistics/Documents/Injury_Facts_43.pdf`; the reader is referred to footnote 6.

you believe me?" Of course, not only does this not lead me to believe in ESP—it actually increases my confidence that he's crazy! (Try it—repeat the calculation, but taking the new beliefs you have about ESP and Artemy as priors.) He provides me what he thinks is more and more evidence for ESP, but my priors mean I take it as evidence for something else.

We've taken an extreme example here, with some things millions of times more likely than others. The basic pattern, however, where people interpret the same evidence and draw opposite conclusions, is common. We'll return to this in Sec. 5 below.

4 John Maynard Keynes and Putting Numbers into Minds

Ten percent; zero point nine; ten to the negative eighteen. To be fluent in Bayes means to be happy putting numbers onto degrees of belief. But where do they come from? We don't make *all* of them up: for example, we derived the tiny degree of belief $P(\text{predict}| \sim \text{ESP})$ to be 7×10^{-31} from saying that the probability of an unbiased (unmagical) coin landing heads was 50%. Meanwhile, the very point of Bayes' theorem is that you get more out than you put in; $P(\text{predict}|\text{ESP})$ is something we derived.

Yet we did have to put some things in, like my prior degree of belief that Artemy is crazy. Where do these numbers come from? If you look in the footnotes, I jokingly relate the degree of belief I have in Artemy being crazy to the likelihood I attribute to getting struck by lighting. And perhaps we do learn something about the strength of beliefs like this, by introspection—this is just about as likely as that; this is less likely; this is more. We spend a lot of time asking ourselves what we believe more, and while we have all sorts of ways of doing that (deductive reasoning, guessing, gut feel) a lot of these activities cash out as comparing and ranking. We can compare and rank numbers, too ("is x bigger than y?"). So perhaps it's not crazy to attach numbers to beliefs, in ways that respect our ranking and ordering, to keep track of exactly what's more likely than what.

But it's not everything, and in particular it's not quantifying. How can we anchor these beliefs—say not just that A is more likely than B ($P(A) > P(B)$), but that $P(A)$ is, say, 0.1? A classic route to putting numbers on beliefs is via to the gambling table. I measure your degree of belief in something by the amount you're willing to wager on it, and the odds you're willing to take. Such a move has been made since the eighteenth century.[6]

[6]Why not before? For a long time, it was remarkable to me that the connection between probability and gambling took so long to be made. The Romans gambled, but didn't compute probabilities? Newton invented the calculus before we knew how to wager on blackjack?

Was it just that, for some philosophical or theological reason having to do with an all-seeing God, humans could not imagine chance before the eighteenth century? Impossible, of course: if anything, they were more acquainted with the *rota fortunae*, the wheel of fortune, than later centuries. So the failure to mathematize probability seemed crazy to me, because a little such knowledge gives you a huge advantage in games of chance. It was as if the human race conspired to leave money on the table. It all seemed crazy, that is, until I looked at the ways in which people actually played games, and looked at the material objects they played them with.

If you do this, you learn that the dice that Roman centurions must have tossed at the foot of the cross were not the machined tools of twentieth century Las Vegas; and the backs of the cards in Carravaggio's *Cardsharps* a far cry from the mechanically reproduced and indistinguishable patterns that anonymize them today. All games of chance, I believe, were biased in unexpected ways by the objects one played them with. One built up knowledge of the biases in the dice, say, over time, and wagered and bet dynamically as one watched others at the game. The tools of gaming were *themselves* subject to both learning (how biased is this die) and negotiation (if the biases can be manipulated, can I maneuver the game in such a way that I get to do it?) Far too complex for probability theory which deals, at least in its most elementary form, with stationary cases where probabilities are fixed, and known, ahead of time. Such conditions did not emerge until the tools to machine such systems were invented. I haven't proven this theory yet, but it's a good one.

You're really sure that something is true? You attach degree of belief 99% to it? Well, if you think it's only 1% likely to be false, then you'll be happy to take a bet where you get $1 if it's true, but you pay me $90 if it's false, because "on average", if we play the game many times, you'll be making nine cents a game (to get nine cents/game, look at it this way: in one hundred games, you'll win $1 about 99 times, and lose $90 once).

There are plenty of reasons why you might not want to take this bet, however, even if you hold that degree of belief. One example is if I find it more painful to lose a lot of money than to gain just a little, even if I win "on average". The relationship can be broken in the other direction, too—I might be happy to wager a small amount for a chance to win big (the lottery ticket story—a one in a billion chance to win a million dollars). We think about people who play the lottery as being irrational, but are they? Say you only have a dollar, and you were starving to death in a ridiculously oppressive country where the cheapest thing is a can of beans that costs $2. Facing immanent starvation, you wouldn't be crazy to bet your dollar in a game with a 1/3 chance of winning $2, even though "on average" you lose. A dollar in that world is worth nothing, but two dollars saves your life.

The complexity of cashing out numbers as actions was brought home to me in discussions concerning the 2016 election (see Section 9 for more). I estimated my degree of belief in a Clinton victory at about 99%; my friend Abe Rutchick then offered me a bet. The resulting discussion is shown in Fig. 1. Some readers might consider me cowardly to reject some of Abe's offers. That's fine, since I don't consider gambling a sign of virtue; indeed, it just goes prove my point because I do, introspectively, hold a very high degree of confidence in Clinton's win, but you can't induce me to place the right kinds of bets to prove it. Not just risk aversion, but the incommensurability of values comes into play: I am (introspectively) willing to pay for an expensive meal with Abe, even though I can't put a dollar value on it. John Searle puts this very nicely, in his 2001 book *Rationality in Action* [2]:

> it seems to be a strict consequence of the axioms [of decision theory, the translating of probabilities into actions] that if I value my life and I value twenty-five cents ... there must be some odds at which I would bet my life against a quarter. I thought about it, and I concluded there are no odds at which I would bet my life against a quarter, and if there were, I would not bet my child's life against a quarter. ... I argued about this with several famous decision theorists, starting with Jimmy Savage in Ann Arbor and including Isaac Levi in New York, and usually, after about half an hour of discussion, they came to the conclusion, "You're just plain irrational." Well, I am not so sure.

One of the many things that makes betting your life on a quarter crazy is that if you lose, there are no more bets for you. This terminates the process unexpectedly and makes averages harder to compute. The more general (and less fatal) case of losing everything—and thus being unable to get back in the game to make your losses back—appears in less extreme circumstances where it further frustrates the thinker and requires careful distinctions between "averages over many players at one time" and "averages over one player over many times"; see my colleage Ole Peters on the classic "St. Petersburg" game [3]. When it comes to unique events, like elections, or even historical facts, where repeated bets can *not* be made, the contortions become even worse.

If actions in the real world can't measure these hidden degrees of belief, are they really real? For many people in the hard sciences it's the only story we have, because if we want to take states of mind seriously (it is felt) we have to describe them mathematically. We go, then, to the laboratory with the quantification we have, not the quantification we want. A Bayesian story might be good enough, or at least a starting point for something better.

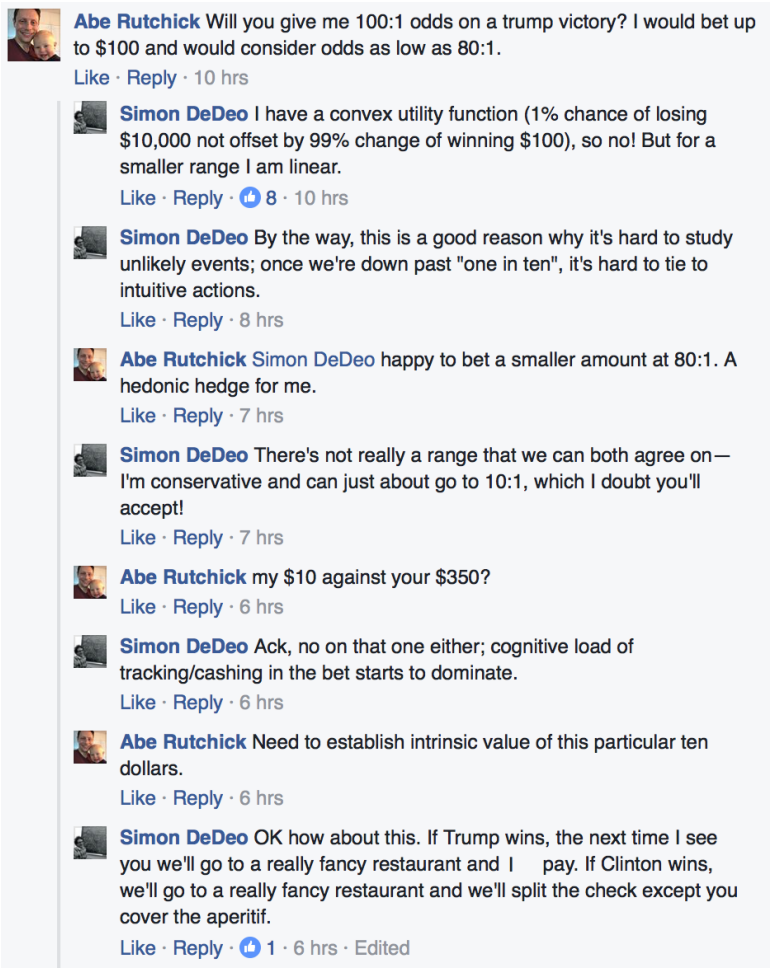


Figure 1: Contrary to what you might learn in an introductory economics class, translating degrees of belief into action is harder than it appears. There are all sorts of reasons why I might believe that an option is only 1% likely to occur, but refuse even to take a 100:1, 80:1, or even 35:1 payout, depending on the amount in play. If we're serious about introspection, it gets harder still—the value of buying my friend a meal seems untranslatable into a dollar amount, meaning that I (Simon) can't think of a strict monetary payoff that would be equivalent to the final deal I offer my friend Abe.

It's worth, however, looking at what happens when people try to go further. John Maynard Keynes, the twentieth-century British economist, was famous for putting beliefs—the cognitive states of participants in the marketplace—at the heart of both his economic theories and his policy recommendations. Businesses could fall into a self-fulfilling pessimism: refusing to spend, firing workers, and thereby, in the aggregate, depriving themselves of the very markets they needed for their goods. Keynes was no stranger to the measurement of mind; in his early career, before becoming an economist, he had studied probability from a philosophical point of view as a model of belief formation; his first book was *A Treatise on Probability*. The stories Keynes told there is far richer than the Bayesian account presented here; in particular, Keynes presented a coherent mathematical system in which the attachment of a number to a probability, or degree of belief, was neither necessary nor even always possible.

Keynes' example, appropriately enough for someone writing in the United Kingdom, concerned the question of whether or not it will rain. A Bayesan can always attach a degree of belief to the proposition that it will rain. If we are very uncertain, then it might be close to 50%, but we can not, coherently, refuse to do so. Any story we tell about the ways in which it could rain will, when phrased in the machinery allow us to extract a number, between zero and unity, that corresponds to the degree of belief we have. Say we think there's a 5% chance that there's a drought, and if there's a drought, then there's only 1% chance it will rain; and then that there's a 95% chance of normal weather, for which the chance of rain is 60%; the overall chance of rain is just over 57%. Keynes disagrees with the idea that this can always be run, and in a widely-quoted passage he writes:

> Is our expectation of rain, when we start out for a walk, always more likely than not, or less likely than not, or as likely as not? I am prepared to argue that on some occasions none of these alternatives hold, and that it will be an arbitrary matter to decide for or against the umbrella. If the barometer is high, but the clouds are black, it is not always rational that one should prevail over the other in our minds, or even that we should balance them, though it will be rational to allow caprice to determine us and to waste no time on the debate. [4]

Though it might pain the Bayesian to hear me say it, there's something right about this. Keynes, like Searle, thought deeply, and if you read carefully here you see that he objects not to the ascription of degrees of belief, but rather to the rationality of doing so. Just as John Searle considers some tradeoffs to be irrational and sense-defying, so does Keynes judge some practices of belief-formation. Hidden in Keynes' objection to the Bayesian account, in other words, is I think a rich *ethics* of thinking. In his youth, as a Cambridge Apostle, Keynes had come to admire the philosophy of G.E. Moore, whose *Principia Ethica* argued that the highest good was an appropriate form of contemplation. No shoving numbers onto sentences for Moore or Keynes! (But no mystification either—Keynes presents an alternate theory where the objects that describe our degrees of belief are more complex yet.)

5 Neutrinos, Cable News, and Aumann's Agreement Theorem

In March 2011, the OPERA collaboration, based in Gran Sasso, Italy, reported a highly unusual discovery: that neutrinos emitted across the border in Switzerland were traveling faster than the speed of light. The effect of this piece of data was electric: the news rocketed around the Internet and then, very soon after, the science sections of the major newspapers in Europe, the United

States, and beyond. And yet, within physics departments, it barely passed without a blip—the fuss was even, I venture to say, a bit embarrassing. Why did the same piece of information induce such different subsequent belief?

The answer goes back to the *priors*—$P(T)$. The Gran Sasso result was in direct violation of the theory of special relativity. Most physicists have very strong beliefs in special relativity; they form these on the basis of both experiments, and also on the general coherence of the theory itself—what William Whewell first called "consilience" [5].[7] Not only does special relativity predict the outcomes of particle scattering experiments in a lab, but it plays a central role in the explanation of phenomena in a vast number of extremely disparate fields of physics. (There it goes by the more sophisticated phrase "invariance under SO(3,1)".) The response of nearly every physicist on the planet was the same: there must be something wrong with the OPERA experiment. (And, indeed, there was—a cable that was not plugged in tightly enough, causing a very slight delay in timing signals that mimic'd the appearance of superluminal velocity.)

By contrast, the science journalism community did not have as strong priors on special relativity. Many (though far from all!) science journalists come out of journalism school, rather than science departments. So the strength of their belief in special relativity is based upon the extent to which they trust, without other evidence, the testimony of a few physicists. When confronted with an enthusiastic PR officer (say) from Gran Sasso, and perhaps a few J-School theories about pulling back the lid on powerful secret interests, the balance of certainty shifted enough to write the articles. (There's an implicit utility function here—the penalty for being wrong, versus missing the scoop—that turns belief into action, but this is the domain of game theory, and that's another article altogether).

Note that—as in the Artemy and ESP case—physicists didn't need to think the OPERA collaboration was particularly dysfunctional. We only needed to believe more strongly in the truth of special relativity. Some physicists (and other observers, of which I count myself one) did not discount the result as much as others—not because we thought the OPERA collaboration was particularly good, nor because we doubted special relativity, but because we wondered if there was some feature of special relativity that we hadn't yet understood, and that would allow us to accommodate the (still revolutionary) Gran Sasso result without losing the coherence of the rest of the theory.

A third example, now that you've gotten the hang of it. Say you turn on the TV and hear (for the first time, cast your mind back) the newscaster say "Obama is a Muslim". What do you believe? In cartoon form—and Bayesian accounts of human cognition always have a cartoony feel—you have a prior about the reliability of the television station, and the possibility that the president of the United States has been maintaining a fictional religious identity in order to conceal a secret agenda. Depending on these priors, rational thought—at least, rational as defined by the criteria on the first paragraph of the first page—will lead two people to very different beliefs, depending on the state of their priors

Note a somewhat sinister aspect here, not present in the previous examples. If it is the case that Obama is a Muslim, and therefore that the president of the United States has a secret religious identity, and nobody else is reporting this, this implies a conspiracy of such vastness that most elite sources of information are probably also implicated. A Bayesian would adjust their beliefs about the reliability of other news organizations downwards, in response; it may not make you believe the news station more, but it will erode your faith in journalism in general.

All three examples go to show that optimal Bayesian reasoning need not lead people to agreement. This is contrary to how we often try to settle arguments, where we may want to say "we

[7] Suggestion for a Ph.D. Thesis: a Bayesian theory of consilience.

disagree; let's all sit down and talk this out"—in the spirit of Leibnitz, "calculemus".[8] As our examples here show, such disagreements may *never* be resolved, if the priors are too different. If the physicist can convince the J-School student not to report the Gran Sasso result (just yet), it is in part because they have gone back to earlier assumptions where the priors are sufficiently strongly shared. (As a contrary example, imagine someone whose priors are—for some reason, say having to do with your sex, race, or religion—that you are fundamentally unreliable. No amount of evidence you provide will ever allow you to overcome even the slightest bias that person holds.)

This is a sad story. Peircean and Popperian accounts of a mystical future agreement among all rational agents *must* supplement Bayes with something—even if only that we, at heart, share some truly fundamental priors.[9]

Oddly enough, if you do make that jump, then *the only* thing we need to do is agree on our priors. An amazing theorem, due to the economist Robert Aumann, says that (1) if two agents are rational (follow the Bayesian rules of this paper), (2) if they have the same priors, and (3) if they have common knowledge about each other's beliefs, then they "cannot agree to disagree" (*i.e.*, they must agree). Put poetically, say I encounter a child from the slums of New Delhi. She and I have had completely different life experiences; we have seen different things, learned different things, and now (because of those experiences) believe different things. If she and I are both rational agents, however, and share the same priors, and *if we discuss long enough*—"hmm, I believe this, but you don't, but now I see that you must believe the opposite because of this, but I don't believe your evidence for that because of this, but I do adjust my belief here what do you think about that?"—we will, according to Aumann's theorem, come to share the same views.

This is true even if we think the other person is an unreliable witness! It works, in other words, even if I hold beliefs about her mis-remembering or mis-reporting evidence. As long as I somehow know what her beliefs are, and she knows mine, and I know she knows mine, and so on, and we agree on our priors, we will come to agree.[10]

I'm not even kidding [6]. In the words of Aumann's abstract, which spends most of its time defining the discussion bit (common knowledge),

> Two people, 1 and 2, are said to have common knowledge of an event E if both know it, 1 knows that 2 knows it, 2 knows that 1 knows is, 1 knows that 2 knows that 1 knows it, and so on. THEOREM. If two people have the same priors, and their posteriors for an event A are common knowledge, then these posteriors are equal.

where "posterior" is just fancy for "the beliefs you have after updating your priors given your (idiosyncratic) experiences"—$P(T|D)$. Better yet, for optimists of the human condition, the computer

[8]The fact that this never really works anyway means, of course, that the application of Bayesian models to human cognition has at least one successful prediction under its belt.

[9]The distinction between a prior and a posterior is a fluid one. Most things I might describe as a prior—such as my belief in the OPERA collaboration's reliability—are secretly beliefs influenced by other forms of evidence, which themselves depend on other priors, which, on further examination turn out to be themselves derived from priors, *ad infinitum*. Practically speaking, we roll with the priors we have, cracking them open when we find ourselves in disagreements with other rational beings. The origin of the priors themselves is mysterious. Examine Eq. 2 again: one never, in the Bayesian framework, intuits beliefs—one only updates them, going from $P(T)$ to $P(T|D)$ to $P(T|D, D')$ to.... Reverse that order. Can it be "evidence all the way down"? The standard answer is no: that priors at some point are fixed by extra-logical forces such as evolution.

[10]Important note: we do have to be honest about our beliefs! We may think the other a scoundrel, but if the theorem is to work, we must respect them enough to tell them so. I am glossing over a hole in my use of Aumann's theorem, which is how we gain knowledge of the other person's beliefs. Aumann's theorem says that once the beliefs become magically known, they will eventually (after discussion) come to be shared. If we are good Bayesians, we'd have to build a theory of our partner's beliefs based on the words they emit and gestures they make when they try to tell us.

scientist Scott Aaronson showed that it may not even take that long [7].

The real problem—trauma—is the need to have priors that agree. What do you believe before you start getting data—before, in other words, your experiences begin? An optimist might say that evolution would give us all similar priors; a pessimist would point out that it would make the subsequent intergalactic war, with aliens that evolved under very different conditions, that much more violent.

The jury is out. Because of the optimality results (page one, paragraph one) Bayesian reasoning is often used in artificial intelligence and machine learning algorithms. Indeed, if we had enough computer power, that's exactly what we'd do; when we don't use Bayes, it's because it can be expensive to compute and the machine takes shortcuts.

But it goes beyond the world of machines, because it also seems to describe human behavior—or, at least, to provide interesting thought experiments that illuminate features of human behavior in a new way. It can explain why the more the person at the café tells you about their ESP experiences, the less and less you believe them; it can explain why rational people can disagree, when given the same data, and even why, as you give them more and more shared experiences, they might draw further and further apart.

It does this explaining by reference to mathematically optimal reasoning, which has a pleasing feel. Simple models seem to show that, even when we're behaving in ways that seem irrational, we may not be so irrational after all. Bayesian accounts of reasoning also suggest that disagreements might be resolved by argument and discussion of an appropriate form, rather than appeals to emotion, or violence. They even tell you some things about what that appropriate form is. It's not a Pollyanna theory, where things can never go wrong, but it begins to specify the conditions under which the course of human history might go right.

6 Specifying Priors and the Zen Koan of Marvin Minsky

The role of priors in Bayesian reasoning is not uncontroversial: by adjusting her priors, a scientist can reverse the implications of an experiment. But this is not as bad as it might seem. In general, there's a sense that as long as priors do not attribute zero probability to the "true" theory, a sufficient number of experiments, of sufficiently wide range, will—eventually—overwhelm priors biased against reality. The scientist who conveniently adjusts her priors will be wrong in the long-run.

As the ESP example shows, though, there are limit cases. It tends to feel, however, as if these limit cases depend for their effect on artificially restricting the space of theories. We do, for example, have many different methods of assessing Artemy's sanity; we can send additional observers; we can conduct new experiments; and a sufficient number of these should, in the final analysis, be sufficient to overwhelm even the strongest biases. If Artemy had reported the results of a test of Bell's Inequality, before I had Quantum Mechanics, I'd be dragging my feet, and maybe even a bit rude, too—but not unconvincable.

Weasel words, however, abound, when we try to answers these questions: "should", "final analysis", "sufficient number". I am aware of no theorem that demonstrates the asymptotic independence of belief from "reasonable" priors. Nor, indeed, do I have a definition of what it means for priors to be reasonable.

Modulo this rather scientific faith, our lack of understanding places us, if provisionally, in a strangely Cartesian place of absolute doubt, where the dependence of our beliefs on our priors

means that vast systems of contradictory beliefs co-exist.[11] Indeed, we are in a place even worse than where Descartes found himself. If we were only uncertain about the world, we would have a well-defined system of beliefs, most of which are around 1/2; but priors are sufficient not only to turn certainty into doubt, but doubt into certainty. (Why not factor this in? If different priors give different beliefs, why not put priors on the priors, and average? In the literature, this is known as a "hyperprior"—and only, of course, punts the problem one step back.)

None of this invalidates the Bayesian project; indeed, one might phrase the result as the *discovery* of priors. The fact that we have been doing statistics without talking, or even knowing, about them doesn't mean they didn't exist. They were there all along, lurking in implicit and incohate form for as long as we had been reasoning: a state of innocence, rather than grace.

An "AI Koan"—a parable, half-joking—from the early days of artificial intelligence research summarizes this position. Attributed to the computer scientist Danny Hillis, it tells the story of two giants of the field, Marvin Minsky, and his student Gerald Sussman, from the early 1960s:

> In the days when Sussman was a novice, Minsky once came to him as he sat hacking at the PDP-6.
>
> "What are you doing?", asked Minsky.
>
> "I am training a randomly wired neural net to play Tic-Tac-Toe" Sussman replied.
>
> "Why is the net wired randomly?", asked Minsky.
>
> "I do not want it to have any preconceptions of how to play", Sussman said.
>
> Minsky then shut his eyes.
>
> "Why do you close your eyes?", Sussman asked his teacher.
>
> "So that the room will be empty."
>
> At that moment, Sussman was enlightened.

"A Selection of AI Koans", *New Hacker's Dictionary*; Ref. [8]

[11] David Deutsch, in *Fabric of Reality*, provides a beautiful example of the scientific faith—including a example of how, by sheer force of intellect, he battles his way out of a "brain in the vat" virtual reality; see Chapter 5.

7 Further Reading

If you're interested in the quantification of belief, and questions like how beliefs change over time, and how beliefs relate to, and influence each other, you will want to learn some information theory. See the companion article, *Information Theory for Intelligent People*, `http://santafe.edu/~simon/it.pdf`.

The physicist E.T. Jaynes died in 1998; his magnum opus, *Probability: the Logic of Science* was reconstructed from his typescript notes (that appear here as Ref. [1]). Rather mathematical, it provides the modern foundation of the use of Bayesian reasoning—and it also provides the ESP example for this article (in the chapter "Queer Uses for Probability Theory"). Rumors suggest that the polemical and harsh nature of the book in parts was due less to Jaynes himself than the passions of his disciples—a Christ and St. Paul scenario.

The physicist David MacKay, while not a Jaynesian, wrote perhaps the best sequel to Jaynes' book, again, highly mathematical, and appropriate for the modern AI era, called *Information Theory, Inference, and Learning Algorithms*; it's available free at `http://www.inference.phy.cam.ac.uk/itprnn/book.pdf`, but also in a lovely hardback. David was a much-loved man of reason, knighted by the British Queen, made a Fellow of the Royal Society, and, when not writing clearly for us, advising the British government on sustainable energy, and raising a family—all by the age of 48, when he died, far too young, of stomach cancer.

The physicist David Deutsch, spot the pattern, has written a great deal on the role—and limits—of Bayesian inference, in popular writings that are just about as bizarre (and fun) as Charles Sanders Peirce (physicist). His books *The Fabric of Reality* and *The Beginning of Infinity* pick up where Karl Popper (not a physicist) left off, and build an entire metaphysics around the use of reliable reason and hypothesis construction.

8 Technical Notes on Working with Probabilites

Once you get the hang of it, Bayesian probability is addictive. The notation for probabilities varies from person to person and field to field, but fortunately there are some general rules. By convention, when we write $P(x, y)$, the probability that x and y are both true, the the event x is drawn from a set of possibilities we label X, and y from a set of possibilities we label Y; X and Y, once we attribute probabilities to their sentences, are often called "random variables".

Each set must *exhaust* all the possible values for that variable, and be *mutually exclusive*: for example, Y could be a trio of sentences, {"the student chosen at random is male", "the student chosen at random is female", "the student chosen at random is neither male nor female"}. "Exhaustive" can be a subtle concept: if, for example, we are working from an archive with missing entries, then Y might require a fourth sentence: "the student's gender is unknown".

Many errors in working with probabilities have their source in ambiguities; most commonly, ambiguities involving the sentences that make up your sets. Consider the following scenario: Cody Zeller[12] is near a Penn State player in the final quarter; the Penn State player trips and falls. You are curious about how the Penn State player being on the ground is related to the possibility that Zeller fouled him.

You might consider having two random variables to describe the situation. One, call it X, might have two sentences, {"Zeller fouled him", "Zeller didn't foul him"}; the other, call it Y, might be {"Penn State guy hasn't tripped", "Penn State guy has tripped"}. Because each of these sets separately exhausts the possibilities, they're good.[13]

Or you might consider one random variable, Z that has four sentences, combining all the different possibilities, such as "'Zeller fouled the guy and the Penn State guy hasn't tripped". But if you tried to reason from a variable Z that had two only two sentences, {"Zeller fouled and the guy tripped", "Zeller didn't foul and the guy tripped"}, you would later find yourself in confusion—it is entirely possible that Zeller did foul the guy, but he caught his balance and didn't trip, and by excluding this from your set, you will get incorrect answers when you grind through the mathematics.

Or, if you had two sentences, {"Zeller fouled", "Zeller fouled and the guy fell"}—in this case, the sentences are not mutually exclusive (both could be true at the same time). Often a sign you've messed up here is that your probabilities for all the items in a random variable don't sum to unity.

One of the trickiest things is how the use of probabilities violates how your math teacher taught you to work with functions. In high school, you might encounter a function $f(x)$, which could be, say, the quadratic $x^2 + x + 1$. You were able to change the argument, so you could write $f(c)$, and everyone knew that you meant $c^2 + c + 1$. The "f" in $f(x)$ or $f(c)$ names that particular functional form.

However, when we write probabilities, we use the same "name", P, for everything. We rely on the variable name to tell us which one; $P(x)$ means the probability of event x; $P(y)$ the probability of event y. But what if we want to use a variable for events? For example, say we want to take the joint probability $P(X, Y)$, and sum over all the possible sentences in X to get the probability of Y

[12] A one-time basketball player for Indiana University, a wonderful institution that, without its knowledge and thanks to the graciousness of the American people and the citizens of Indiana, paid for me to write this.

[13] Here you know they do because of something pretentiously called the "law of the excluded middle"—which says that "either x is true, or not-x is true." Once something obvious is called a law, you have the benefit of being able to ask what happens if the law isn't true. Some (to this author tendentious) interpretations of quantum mechanics make use of novel logical systems that violate the law of excluded middle. In a different vein, the logicians Graham Priest and Richard Routley proposed to reject this law in the classical case to construct a new "dialetheic" logic (see, *e.g.*, Priest's *Doubt Truth to Be a Liar*). The extensions of such a system to encompass doubt and uncertainty remain to be done, but would require revision of the basics of probability theory itself.

alone; we might write

$$P(Y) = \sum_{i \in X} P(i, Y), \tag{9}$$

where $i \in X$ is shorthand for "all of the different sentences in X", and i is a variable. In the formula $P(i, Y)$, it can be easy to forget what the first argument means; it's often good to cue in a reader by writing $P(X = i, Y)$. This gets particularly important when the sentences are naturally written as numbers (*e.g.*, "the price of Apple is \$16/share")—if you have a joint probability describing two stocks, you can write $P(12, 19)$, but it might be better to be careful and write $P(\text{Apple} = 12, \text{IBM} = 19)$. We'll see this in the next section, where we attack a harder problem about waiting for the bus.

9 Nate Silver, Sam Wang, and who will win the election?

Update: please see "Wrong for the Wrong Reasons", `http://santafe.edu/~simon/wrong.pdf` for a postmortem of 2016 election polling and statistics.

In 2004, the Princeton neuroscientist Sam Wang began applying Bayesian tools to the prediction of U.S. presidential elections. Polling companies—the people who called up voters, or did internet surveys, or worked by mail or knocking on doors—had begun putting their raw data online. If you wrote down models for how people's underlying voting habits would be reflected in these polls, then the same models would also predict who would win the elections. Sam built some simple models (or, rather, wrote down some nice stories about $P(D|T)$, where T were the underlying voter preferences, and D was the polling data), and predicted the outcome of the 2004 presidential election on the nose.

In 2008, Sam was joined by Nate Silver; Nate had learned the power of Bayesian tools as a sports statistician.[14] Together with a number of other hackers and part-timers, they changed the face of election prediction, taking it out of the hands of the pundits—who often inflated uncertainties in order to create a television narrative—and putting it back into the hands of the people. This new kind of psephology, with Bayes at the heart, is more accurate than anything we've had before, helping wonks and nerds, and ordinary citizens, focus their attention on what matters and where the most is at stake.

How do Nate and Sam do it? Let's take some steps towards a Bayesian model of election prediction. In doing so we'll get a few toy examples of how Bayesian statistics is actually used in practice. Let's assume (for the moment) that we have good ways to build beliefs about which candidate will win state-by-state. Then a natural thing to do is to combine these to get the degree of belief one should have in a particular combination of outcomes. Say, for example, we want to know the probability that Clinton wins both Pennsylvania (PA) and Ohio (OH), and we have polling data (D_{PA}, and D_{OH}). It's then natural to write

$$P(PA \;\&\; OH|D_{PA}, D_{OH}) = P(PA|D_{PA})P(OH|D_{OH}) \tag{10}$$

where (for notional simplicity) the arguments of the Ps will refer to Clinton (*e.g.*, $P(PA|D_{PA})$ means "the degree of belief I have in Clinton winning Pennsylvania, given polling data from Pennsylvania D_{PA}"). This is the equivalent move to saying that the probability of getting Heads from two coins is just the probability of getting Heads with coin one, times the probability getting Heads with coin two.

There are three things wrong with doing this. The first is that I haven't told you how to get $P(PA|D_{PA})$ or $P(OH|D_{OH})$. The second is that (as hinted), we're assuming that, giving polling data, states are independent.

The third is that if we try to scale this method up, we need to compute the probability of every single pattern of winning and losing. When we start doing that we realize that we end up with at least 2^{51} different things to calculate, because we have to consider every single pattern (Clinton does/doesn't win OH, Clinton does/doesn't win PA, Clinton does/doesn't win NV, ...) of the electoral college. Every state (and D.C.) contributes two possibilities—either Clinton wins all the electoral votes (EVs) for the state or not; we'll neglect for simplicity states like Maine where they hand out the EVs separately. Even if we can compute one probability every microsecond (a

[14] I'm pretty sure that I overheard Nate explaining Bayes' theorem while waiting for the 55 bus in Chicago one very cold night in 2008, but that's a story for Section 10.

standard lag time for code like Python), it will take seventy-one years to compute them all, and 2,000 Terabytes just to store the results.

Begin with this third problem, which turns out to be the simplest to solve. On reflection, there are three ways out.

You could say, well it's not the case that we have to consider every possibility—there are many states that Clinton is practically guaranteed to win, so we shouldn't bother computing things like "the probability that Clinton loses New York but wins Ohio". She won't lose New York. If there are just ten swing states, we can get by with 2^{10} different combinations—about a thousand—and that's pretty quick. Or (a second idea): we can simulate an election, going through the 51 electoral votes and tossing biased coins for each state to get a sample of what might have happened—a kind of *SimCity*, or Sid Meier's *Civilization*, writ small. If we do this lots of times, and keep track of the final votes, we can build up a probability distribution for anything we care about (*e.g.*, and most simply—we can simulate the election 10,000 times and see how many times Clinton wins). This second method is a very simple example of what the stats gurus call "Monte Carlo" (after the famous gambling city in Monaco). Monte Carlo solves a Bayesian reasoning problem by doing what all Bayesians secretly fantasize about doing: make it into a (very controlled example of a) frequentist sampling story. Monte Carlo is pretty good because (roughly speaking) if you run 10,000 simulations and a particular outcome never occurs, your degree of belief in that outcome happening in the real world should be no more than around 10^{-4}.

A third method is a little trick that Sam Wang is particularly proud of [9]; it turns out that if all you care about is the number of electoral votes—and not where they come from—you can get that really quickly. First, write down the polynomial

$$f(x) = \prod_{s=1}^{51} [(1 - P(s|D_s)) + P(s|D_s)x^{EV_s}], \tag{11}$$

where s is a variable for the states, EV_s is the number of electoral votes coming from state s, $P(s|D_s)$ is the probability of Clinton winning state s (given the polling data D_s). What's x? Don't worry about it, just write out Eq. 11 blindly, like a robot with no soul (*i.e.*, focus on the syntax of the formula).

Sam's trick is to note that the coefficient of x^N in the function $f(x)$ is equal to the probability that Clinton wins N electoral votes. That only requires you to compute a polynomial by multiplying things together 51 times! If you're a math geek, you love this and can see why it's true; if not, try writing out the product in full for a simple case with, say, four states, and you can see that all the cross-multiplications you're doing amount to considering the 2^4 different possibilities, except that as you add together terms that have the same power of x, you're no longer keeping track of them separately. This is a simple example of something the gurus call a generating function.

Looking at the simplest problem is a nice way to get a sense of the "technology" behind Bayesian computations. You can get an exact answer to a simpler question by hand; you can approximate the full question with Monte Carlo simulations and an expensive computer; or you can be super-clever (sometimes) and answer part of the question exactly and everyone's impressed.[15]

The second hardest problem is the independence one. Is it really the case that "Clinton winning Ohio" and "Clinton winning Pennsylvania" are independent events? In one sense, of course not. If Clinton is losing Pennsylvania badly, she's likely to be losing Ohio, too. But that's not what Eq. 10

[15] In graduate school, these were associated for me with different nationalities. The first method seemed British (pragmatic compromise), the second rather American (throw money at it); the third very Soviet Russia (uncompromising hard-slog brainpower).

is saying, because $P(OH|D_{OH})$ is the probability of Clinton winning Ohio *given* the polling data. If Clinton is losing Pennsylvania badly, then hopefully the reasons that she's losing Pennsylvania badly, and that mean she'll also lose Ohio, will show up in the Ohio polling data. Under this assumption, the Ohio polling data will move up and down with the Pennsylvania polling data. But once I know the Ohio polling data, any further motion in Pennsylvania will be irrelevant to predicting Ohio.

An analogy would be the example of two friends who go to a bar together, and drive home separately. Their drinking is correlated—if Alice starts drinking a lot, then Bob is likely to join her. This means, therefore, that their probabilities of getting into a car accident are not independent: the probability that Alice and Bob *both* crash their cars is higher than the probability that Alice crashes times the probability that Bob crashes. But once you tell me the blood alcohol content (BAC) of Alice, that's all I need to know to predict the chance that she'll crash her car. Formally, it's the case that Alice and Bob are strongly correlated

$$P(A, B) \neq P(A)P(B) \tag{12}$$

but become independent once you know their BACs,

$$P(A, B|BAC_A, BAC_B) = P(A|BAC_A)P(B|BAC_B). \tag{13}$$

The pretentious way to say this is that sometimes variables are conditionally independent. When you build models of many variables, and specify the various patterns of conditional independence, you're well on your way to building what's called a Bayesian network, which is related to what Judea Pearl calls a Causal Network, and is the basis of a beautiful and complex and (in this author's opinion) flawed theory of measuring causality from empirical data, but that's another story.

If Pennsylvania and Ohio going to the polls are like Alice and Bob drinking after work, then Sam's independence assumption does very well. But you can imagine ways in which it goes wrong. The first thing you might say is that perhaps polling data from Pennsylvania really does tell you something about Ohio that Ohio data doesn't tell you. That turns out not to be so interesting, because we can recover independence now just by folding the Pennsylvania data to our Ohio prediction: formally, figuring out how to write down $P(OH|D_{OH}, D_{PA})$.

To *really* break independence, you need some causal relationship that can't be captured by polling. Imagine, for example, that bad news for Clinton in Pennsylvania depresses Clinton-voter turnout in Ohio (perhaps easier if you think about, say, Colorado, where timezones mean the Coloradans can watch the exit polls from Pennsylvania and decide whether or not to vote).

A more abstract possibility is that of *correlated errors*. In order to compute $P(PA|D_{PA})$ we have to turn polls of individuals into predictions about state-wide events. To do this, we have some theories—for example, we might decide that women are more likely to vote, and so $P(PA)$ is higher than you'd expect because women favor Clinton. What if those theories are wrong?

To continue our example, say that we're wrong about women having high turnout. In that case, we not only over-estimate the probability of $P(PA|D_{PA})$, but we're wrong about every state. To take this into consideration, we have no choice but to model everything together, including different theories about turnout, and writing something like,

$$\sum_{T_w=1,2...}^{N} P(PA, OH|D_{PA}, D_{OH}, T_w)P(T_w) \tag{14}$$

where T_w is a (discrete) variable representing N different theories about turnout. Now the outcomes in Pennsylvania and Ohio are correlated again, because they both depend on the value of T_w. To

go back to Alice and Bob—allow me this slightly weird example, it's late at night before the 2016 election—imagine that they're related to each other. Because they're related, they might both have genetic sensitivities to alcohol. If they do, even low BAC levels lead to significant impairment. Now all of a sudden, their futures become correlated again, even once we know their BACs; if Bob has low BAC but crashes his car, then we say "oh no—now that I know Bob just crashed with low BAC, I bet Alice has the same bad gene."[16] The big debates among the stats gurus at the end of the 2016 election center around this possibility: Nate says you have to consider it, Sam says it doesn't matter.

Talking about turnout models leads us, finally, to the question of how to turn polling data into "state win probability data"—the $P(PA|D_{PA})$ that form the basis of the final prediction. This is, in a sense, where all the magic is. In general, Nate has very complex models, while Sam has very simple ones. Indeed, in Sam's 2012 paper, he reports a rather heuristic method that (wait for it) involves taking the medians of all the polling data for the state, and pretending that that median is normally distributed with a variance σ given by the median of all the pollster-reported errors, divided by $\sqrt{N}$. There's some justification for all of these steps, which in the end give you a $P(PA|D_{PA})$, but it's not a simple Bayesian story any more—at best, it's a sort of heuristic approximation to the real story you *should* be telling. If you're a statistical type reading Sam's report of what he does you sort of nod along, half-filling in the gaps as to why each of these steps is a good approximation, but it's not the world's most rigorous demonstration.[17]

Nate has much more complicated models, that involve building careful stories about voter turnout, pollster biases, and so forth. The short version of what this all means is that Sam tends to be much more certain than Nate in his predictions. For Sam, once you write down the state-by-

[16] A common genetic defect like this does indeed exist. It prevents the creation of alcohol dehydrogenase, the enzyme that processes alcohol. Without alcohol dehydrogenase, drinking even small amounts of alcohol leads to the buildup of toxic byproducts that cause nausea, mental confusion and blurred vision—instead of a pleasant "buzz". But is this defective gene "bad"? Actually, alcohol is a terrible thing for your reproductive success, because drunk people make bad decisions. If you could evolve a desire never to drink it, your descendants would do better. Evolution works on slow timescales, but it has had time to act here as well. The first cultures to cultivate a grain easily fermented into alcohol were in Asia, where it first became possible to make rice wine. Today, defective alcohol dehydrogenase genes are preferentially found in East Asian populations, including the Han Chinese, which also have lower rates of alcohol abuse. College students with ancestry from other parts of the world refer to this condition as "Asian flush", perhaps unaware that the joke's on them. The relevant timescales (10,000 or so years since rice cultivation) are just about the shortest that we know evolution can work on in human populations—which is why later wheat cultivation hasn't had time to flush it out of Europe (yet).

[17] Savvy appreciators of Sam's analysis will see how much I've left out. To begin with, Sam now *does* have a model to account for correlations in polls: what he calls the meta-margin (MM), which describes the possibility of a uniform bias across all polls in the direction of one candidate or the other. Hypotheses about what this meta-margin is going to do lead to correlated changes in $P(s|D_s, MM)$, which he can then turn into EV predictions using the generating function trick, $f(x)$.

Sam's tried a few theories about the MM over the years, but in the most recent election, 2016, he used past data to formulate a prior that assumed a symmetric distribution of the meta-margin about zero—*i.e.*, while the polls will, in the end, be biased in one direction or another, he remains agnostic about which direction they'll be biased in. To build his final predictions, he combines a Monte Carlo method with his generating function trick: he samples from the prior distribution over MM (the simulation step), gets a bias, feeds that into $P(s|D_s, MM)$, and gets an EV prediction. He repeats this a few thousand times to produce a distribution over different EV totals, and uses that to get probabilities on which candidate will win. The MM prior reduces the certainty of the final result, but a quick analysis suggests that it can never actually flip the prediction of who is most likely to win the election. At other points in his analysis, Sam has used the MM not to model polling bias, but rather opinion drift in the underlying preferences of likely voters. It's a bit of a hack, but assuming a prior distribution over MM can be thought of not as a theory about polling errors, but as a story about actual voters randomly flipping their opinions back and forth in a noisy, but correlated, fashion from moment. I thank Joel Erickson for delving into Sam's 2016 methodology with me.

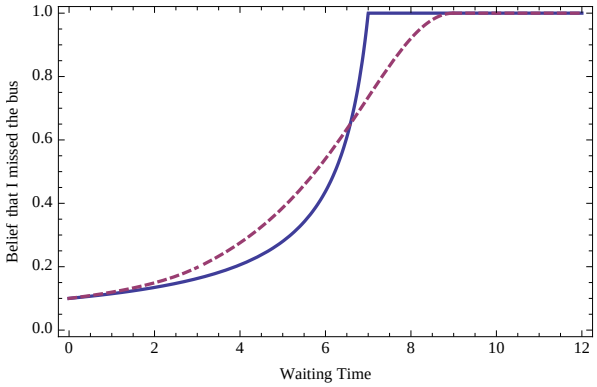
state probabilities, it's pretty much a foregone conclusion, because even slight leans in each state accumulate in favor of a candidate. All you need are enough polls to make those leans apparent enough, and near the end of an election, that tends to be the case.

For Nate, it's very different. Correlations between states mean that a single hidden variable (say, young female turnout, variables like T_w) can wash across multiple states at once, erasing mild leans that favor one candidate. If you're uncertain about the state of those hidden variables, you're much less certain about the state of the election as a whole.

Who does better? Both Sam and Nate have logged great successes, beginning in 2004 when Sam's raw code predicted not just the election, but the exact EV split using his generating function $f(x)$. It's also worth noting that both Sam and Nate have had forehead-smacking moments when they step in to override the results of their more naive models. Sam, for example, erred in 2004 when he "added" likelihood to Kerry to handle what he believed would be late-breaking voters for the Democratic candidate John Kerry—some wishful thinking natural to a committed Democrat.

Meanwhile, Nate stumbled in 2016 when, in a more complex series of events, he failed to predict the nomination of Donald Trump. Prediction markets—*i.e.*, sites that allow punters to place bets on election outcomes—had favored Trump for many months before he won the nomination. But while the clever aggregation of lots of individual beliefs of many thousands of election junkies did very well, the pundits mostly "could not believe it", even as the evidence, at least as perceived by the wisdom of the crowd, piled up. Nate wrote a nice *mea culpa* on how he got caught up in the pundit's game, letting his prejudices and gut feels take over from the mathematics.

(a)



(b)

Figure 2: **Left panel**: the Number 55, feared and loved by generations of students at the University of Chicago, as their only connection to the rest of the city. When will it come? Image from NBC Chicago, 26 August 2013. **Right panel**: Did I miss the bus? What a Bayesian angel believes after waiting for t minutes, given two different theories she might have about bus schedules.

10 Waiting for the Number 55 bus

In this (advanced) appendix, we'll use a little mathematics to show how a Bayesian angel waits for the bus.

For concreteness, we'll take a scholarly example. The University of Chicago, in Hyde Park on the South side of Chicago, is singularly ill-connected to the rest of the city. Students are forced to rely on "the 55", a city bus that picks up along 55th Street and connects a few miles away to the subway system. Returning home late at night, students wait outside the station for the ride that takes them home. But when will it arrive? When temperatures drop, as they do in Chicago, below zero degrees Fahrenheit, the question becomes an urgent matter of psychology.

Intuitively, you don't expect the bus to be there right away. Though it might come quickly, you assume you'll have to wait a bit before it shows up. However, as the minutes tick by, a mounting despair takes over. Is the traffic bad? Did the bus break down? Has the schedule gone irregular? Has the last bus of the night come and gone?

From a Bayesian perspective, a student waiting for the bus is not just waiting for the bus: she is also, simply by waiting there, gathering information for and against different hypotheses about what is happening elsewhere in the city. Each hypothesis, understood separately, predicts a different arrival time. As the student waits, and some of these times pass without a bus, hypotheses are killed off. The information the student gets simply from waiting means that her beliefs about when the bus will arrive—how much longer she has to wait—are in flux.

Let's see how this works in detail. In particular, let's plot out the changing beliefs that the student has as she waits longer and longer. To work through this problem in a Bayesian fashion, we'll do two things. First, we'll describe the space of *theories* the student can have. Each theory will imply a certain probability that the bus will have already arrived while the student was waiting. As the student continues to wait for the bus, her belief in some theories will rise, and her belief in others will fall.

Some theories—such as the theory that buses have stopped running for the night—will be simple to describe. Others will, as we'll see, be a little more complicated, and have some hidden knobs and dials or what are known as "adjustable parameters". Once we've defined our theories, we'll describe the predictions the theories make.

For simplicity, we'll have two theories in our list T: theory a, that the buses have stopped running for the night; and theory b, that the buses run on a regular schedule, spaced exactly seven minutes apart.

Notice that b is not quite as simple a theory as theory a; the buses may be running on a regular schedule, but the student doesn't know whether one just arrived right before she did. So for the student's point of view, theory b has an adjustable parameter, call it ϕ, the number of minutes before she arrived that the previous bus left. By stipulation, ϕ is between zero and seven. If ϕ is zero, the student arrived just after the bus left (argh!).

As for the data, or list of possible observations D, we'll describe it by a single number, t, the number of minutes the student has been waiting; at any point in time, we can ask "what's the probability that you're still waiting after time t, given theory $t \in T$". (You could imagine adding in additional information, such as the observation of buses going the opposite direction or the presence of other people waiting at the stop, that might bear on the truth of different theories, but we'll keep it simple for now.)

Predictions with theory a are simple. If the buses have stopped running for the night, the student will necessarily still be waiting, so $P(t|a)$ is equal to unity (unless t gets really large, let's say, and morning rolls around and the buses start again—but forget this).

The probability that the student is still waiting if theory b is true is more complicated, since b has an adjustable parameter, ϕ ("how recently the last bus came"). We know that $P(t|b,\phi)$ is equal to unity if $\phi + t$ is less than seven, and zero otherwise; informally, if it's been less than seven minutes since the last bus came, she's definitely still waiting; otherwise, she's definitely not.

What should we believe about ϕ? If we're optimists, then ϕ is close to seven; if we're pessimists ("I'm so cold and I'm SO SURE we just missed it I hate my life"), we think it's close to zero. Let's be neutral on the question, and assume our prior on ϕ is uniform: $P(\phi)$ is the same for all values of ϕ. This is a bit tricky because in general ϕ is a continuous quantity, not a list of sentences; we could restrict ϕ to be an integer, and do sums, but for fun, let's use some calculus to do the limit where ϕ can take on any real value between zero and seven and sums over values are replaced by integrals. Then we have

$$P(t|b) = \int_0^7 P(t|b,\phi)P(\phi)\ d\phi. \tag{15}$$

If ϕ is in units of minutes, then $P(\phi)$ will be a constant, c, which turns out to be equal to $1/7$, as can be seen from the normalization condition

$$\int_0^7 P(\phi)\ d\phi = \int_0^7 c\ d\phi = 1. \tag{16}$$

Because $P(t|b,\phi)$ is equal to unity only when $t + \phi$ is less than seven, we have that $P(t|b)$ is equal to zero when t is greater than seven, and

$$P(t|b) = \frac{1}{7}(7 - t) \tag{17}$$

otherwise. As t gets larger, the range of ϕ for which the integrand of Eq. 15 is non-zero gets smaller and smaller.

We specified a prior on ϕ for the parameter of our theory b; now it remains to us to specify priors over a and b as a whole. If we feed $P(a)$ equal to 0.1 (a small chance that the buses have stopped running) and $P(b)$ equal to $1 - P(a)$ (*i.e.*, these are the only two possible theories that could be true), we can now describe anything we desire about the student. For example, we can

use Bayes' rule to flip $P(t|a)$ around into $P(a|t)$, the belief the student has that, given she's been waiting t minutes, the bus is never going to come:

$$P(a|t) = \frac{P(t|a)P(a)}{P(t|a)P(a) + P(t|b)P(b)} \tag{18}$$

This is the solid blue line in Fig. 10. As time rolls by, the student becomes increasingly confident that she has, indeed, missed the last bus of the night. Once the seven minute mark has passed, she knows that all is lost.

This is a nice start on a description of the student's state of belief, but it's troubling in part because of that absolute certainty. Are you *sure*, we might ask, that the bus won't come at seven minutes and five seconds? Perhaps you were mistaken about how often the buses run—was it every seven minutes, or every eight minutes? Let's imagine that the student enlarges her theory b to allow for the possibility that the buses are not spaced every seven minutes, but that the spacing could be anywhere between three minutes and nine (as in "the buses run regularly, every three to nine minutes"). We can add a new parameter to b, s, the spacing, to get we have $P(t|b, \phi, s)$, the probability that we're still waiting at time t if the last bus left ϕ minutes before we arrived and the buses arrive once every s minutes. With a little thought, we can integrate out the ϕ to get

$$P(t|b, s) = \frac{1}{s}(s - t), \tag{19}$$

where s is the spacing, and $P(t|b, s)$ is zero when t is greater than s. We now just need a prior on s, which we'll take to be uniform between three and nine; $P(s)$ is a constant, in other words, equal to $1/6$ (by a reasoning process analogous to how we got $P(\phi)$ equal to $1/7$ when s was seven). Let's do the integral

$$P(t|b) = \int_3^9 P(t|b, s)P(s)\ ds \tag{20}$$

When t is less than three, any spacing is allowable, and we have

$$P(t|b) = \frac{1}{6}\int_3^9 \frac{1}{s}(s - t)\ ds = \frac{1}{6}\left(s - t\log(s)\right)|_3^9 = \frac{1}{6}(6 - 3\log 3) \tag{21}$$

When t is greater than three, but less than nine, some spacings, with s less than t, are no longer possible, and so we have to be careful to include only the allowable ones, for which $P(t|b, s)$ is non-zero,

$$P(t|b) = \frac{1}{6}\int_t^9 \frac{1}{s}(s - t)\ ds = \frac{1}{6}\left(s - t\log(s)\right)|_t^9 = \frac{1}{6}(9 - t - t\log\frac{9}{t}), \tag{22}$$

and when t is greater than nine, $P(t|b)$ is zero (we're doomed again—you can't have been waiting that long if the buses were still running). The resulting $P(a|t)$ that comes from this enlarged theory space is shown as the dashed red line in Fig. 10. Notice that at small waiting times, the possibility that the bus could have come as quickly as three minutes, even if it had just left, makes us nervous: we get more worried that theory a might be true. That's the red dashed line being higher than the blue solid line. This cross-over depends on exactly what we think about the possible range of arrival gaps.

Note that even in this enlarged theory space, we become completely certain that the buses have stopped at some point—now at nine minutes, not seven. A more realistic theory yet might put non-uniform prior on s, such that $P(s)$ was high for "normal" gaps, like three minutes or seven minutes

or nine minutes, and lower (but not zero) for longer ones. One could add in a delay probability, to cover traffic jams or bus drivers getting lost or having to throw drunkards off the bus—left as an exercise to the reader.

One can extract more from these equations. For example, you can think about writing down $P(\phi|t, b, s)$, or, in words, what's the probability that the most recent bus left ϕ minutes before I arrived, given that I've been waiting for time t, and the buses are still coming with frequency s (theory b is true). You can combine this with priors on s to answer questions like "what's the probability that the bus will arrive in the next t' minutes, if the buses are still running":

$$P(t') = \int_3^9 P(\phi = s - (t + t')|t, b, s)P(s)\ ds. \tag{23}$$

Take some time to work through this equation. In words, the formula in the integrand means "what's the probability (degree of belief) that $\phi + t + t'$ is equal to the spacing s, given that I've been waiting for time t", which is then weighted by the prior belief in that spacing, $P(s)$. Once you get handy with conditional probabilities and Bayes rule, it's "simple" to write down answers to questions like these.

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