Information in Complex Systems

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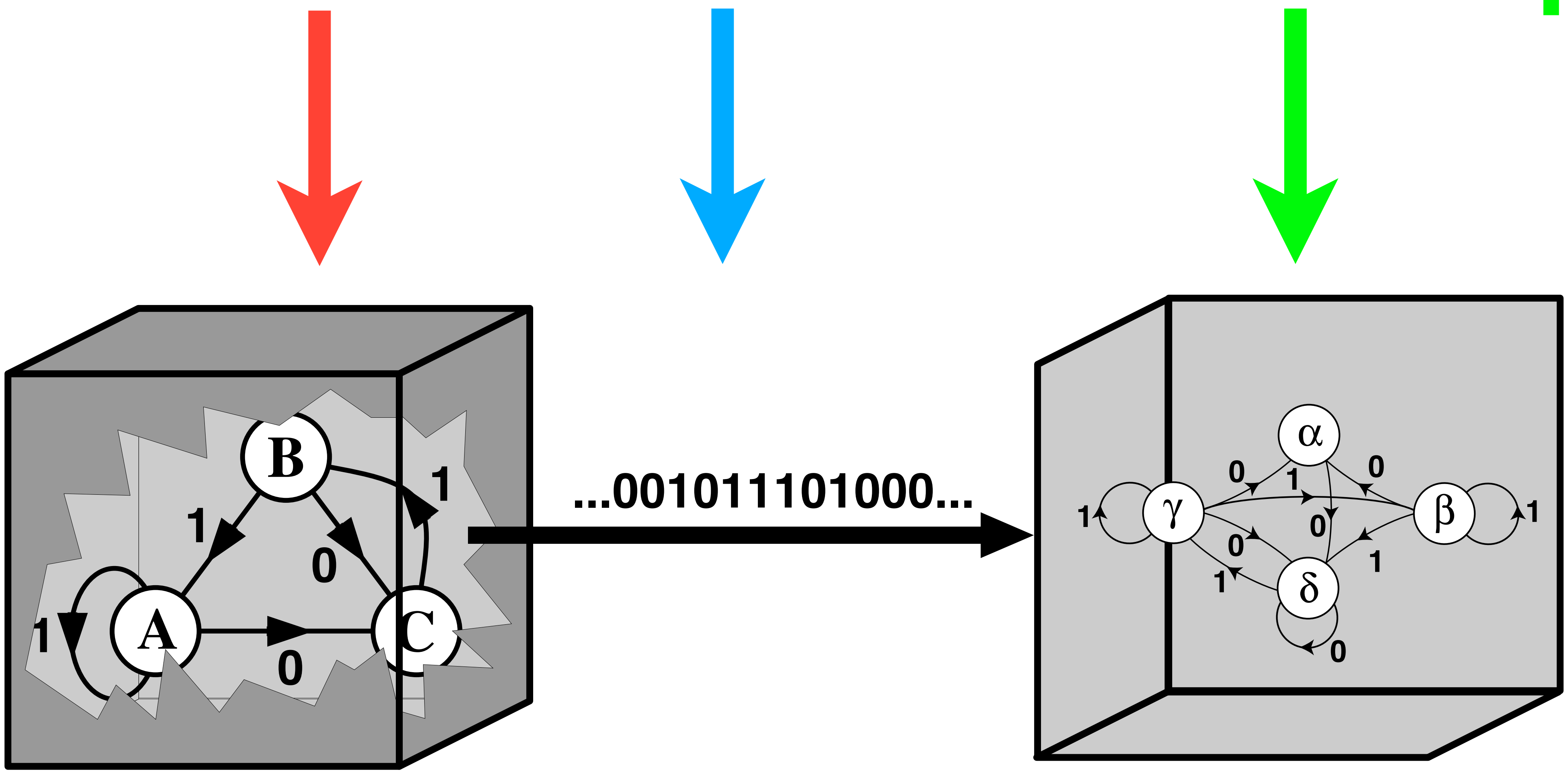
Main Question

Randomness versus Structure?

Now Next Finish Up



System **Instrument** **Process** **Modeller**

The Learning Channel

Information in Complex Systems

Now:

- Algorithmic Basis of Probability
- Information Theory
- Information Measures

Later:

- Measuring Structure
- Intrinsic Computation
- Optimal Models
- Physics of Information

Information in Complex Systems

References? For example:

Stanislaw Lem, *Chance and Order*, New Yorker **59** (1984) 88-98.

T. Cover and J. Thomas, *Elements of Information Theory*, Wiley, Second Edition (2006) Chapters 1 - 7.

M. Li and P.M.B. Vitanyi, *An Introduction to Kolmogorov Complexity and its Applications*, Springer, New York (1993).

J. P. Crutchfield and D. P. Feldman, "Regularities Unseen, Randomness Observed: Levels of Entropy Convergence", CHAOS **13**:1 (2003) 25-54.

R. G. James, C. J. Ellison, and J. P. Crutchfield, "Anatomy of a Bit: Information in a Time Series Observation", CHAOS **21**:1 (2011) 037109.

J. P. Crutchfield, "Between Order and Chaos", Nature Physics **8** (January 2012) 17-24.

A. B. Boyd and J. P. Crutchfield, "Demon Dynamics: Deterministic Chaos, the Szilard Map, and the Intelligence of Thermodynamic Systems", Physical Review Letters 116 (2016) 190601.

See http://csc.ucdavis.edu/~cmg/

See online course: http://csc.ucdavis.edu/~chaos/courses/ncaso/

Processes and Their Models ...

Main questions now:

- How do we characterize measured processes?
  - Degrees of unpredictability & randomness?
  - Use probabilities?
  - What correlational structure is there?
- How do we build a model from the process itself?
- How much can we reconstruct about the hidden internal dynamics?

Algorithmic Basis of Probability

Kolmogorov-Chaitin Complexity Theory

The question:

Algorithmic foundation for probability?

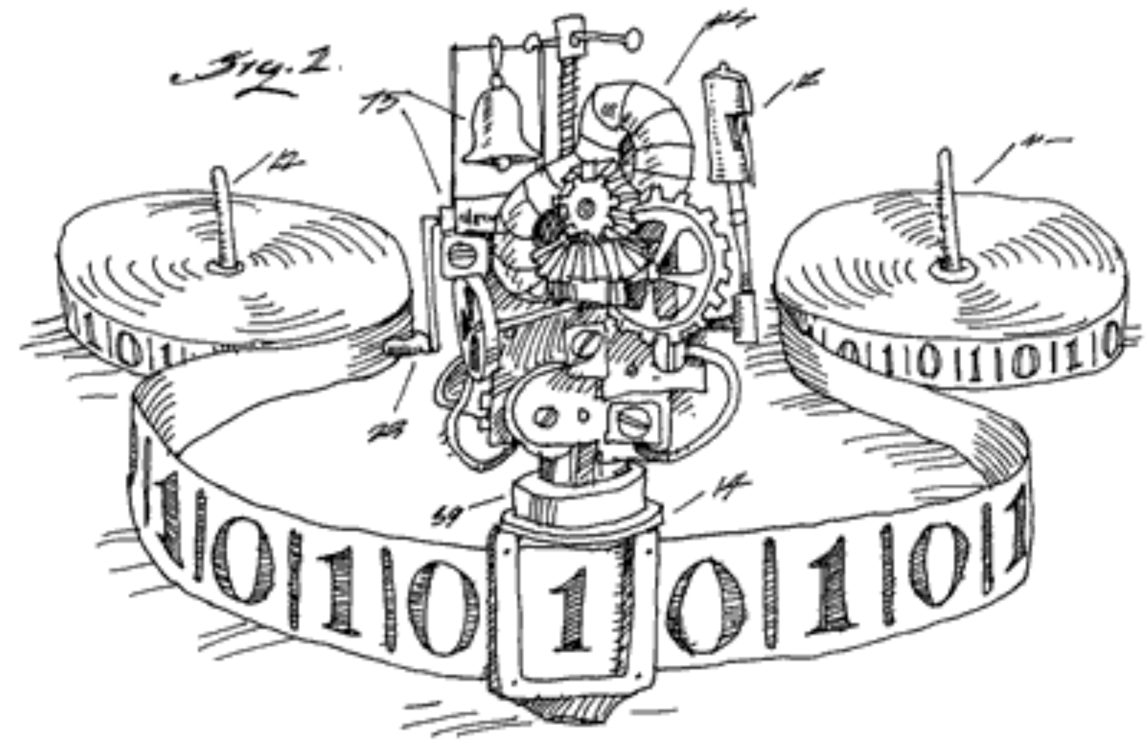
History:

1776: Treatise on probability theory (Laplace)
1920s: Frequency stability (von Mises)
1930s: Foundations of probability theory (Kolmogorov)
1940s: Information theory (Shannon … Szilard 1920s!)
1940s: Automata & computing theory (Turing)
1960s: Algorithmic Complexity Theory
(Kolomogorov, Chaitin, Solomonoff, ...)

Kolmogorov-Chaitin Complexity

Turing's machine (1937):

Finite-state controller +
Infinite read-write tape



Machine M:
Device to generate output $x = 1010111\ldots$ from program p:

$$M(p) = x$$

Kolmogorov-Chaitin Complexity

Universal Turing Machine: U
Sufficient states, control logic, and tape alphabet
$\Rightarrow$ Calculate any input-output function

UTM programs generate output: $U(p) = x$

(Python interpreter w/ infinite memory.)

Kolmogorov-Chaitin Complexity of given object:
Size of smallest program p that generates object x

$$K(x) = \min\{|p| : \ U(p) = x\}$$

Kolmogorov-Chaitin Complexity

Consider Python program:

```
def generate_x():
        print x
```

And so:

$$K(x) \leq |x| + \text{constant}$$

For most objects:

$$K(x) \approx |x|$$

Kolmogorov-Chaitin Complexity is not computable.

(Theorem: No program can calculate $K(x)$.)

Kolmogorov-Chaitin Complexity

Exercise! Which has high, which low $K(x)$?

$$
\begin{array}{l}
00100100001111110110101010001000 \\
10000101101000110000100011010011 \\
00010011000110011000101000101110 \\
00000011011100000111001101000100
\end{array}
$$

π

Algorithm $\Rightarrow$ low $K(x)$

(Bailey–Borwein–Plouffe 1997)

$$
\begin{array}{l}
10000010100011011111101110011100 \\
01101101001100010110010001010100 \\
00101100011011000110001110111000 \\
10110100010000111000111001110011
\end{array}
$$

Random

High $K(x)$

Kolmogorov-Chaitin Complexity

Lessons:

A random object is its own shortest description.

$K(x)$ maximized by random objects.

Probability of objects:

$$\Pr(x) \approx 2^{-K(x)}$$

Alternatives?

Computable?
Scientifically applicable?

Information!

Information ...

Information as uncertainty and surprise:

Observe something unexpected:
Gain information

Bateson: "A difference that makes a difference"

Information ...

Sources of Information?

Apparent randomness:

- Uncontrolled initial conditions
- Actively generated: Deterministic chaos

Hidden regularity:

- Ignorance of forces
- Limited capacity to model structure

Information ...

Information as uncertainty and surprise ...

How to formalize?

Shannon's approach:

A measure of surprise.

Connection with Boltzmann's thermodynamic entropy

Self-information of an event $\propto -\log \Pr(\text{event})$.

Predictable: No surprise $-\log 1 = 0$

Completely unpredictable: Maximally surprised

$$-\log \frac{1}{\text{Number of Events}} = \log(\text{Number of Events})$$

Information ...

Shannon Entropy: $X \sim P$

$$x \in \mathcal{X} = \{1, 2, \ldots, k\}$$

$$P = \{\Pr(X = 1), \Pr(X = 2), \ldots\}$$

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log_2 p(x)$$

Note: $0 \log 0 = 0$

Units:

Log base 2: $H(X) = [\text{bits}]$

Natural log: $H(X) = [\text{nats}]$

Properties:

1. Positivity: $H(X) \geq 0$
2. Predictive: $H(X) = 0 \;\Leftrightarrow\; p(x) = 1$ for one and only one x
3. Random: $H(X) = \log_2 k \;\Leftrightarrow\; p(x) = U(x) = 1/k$

Information ...

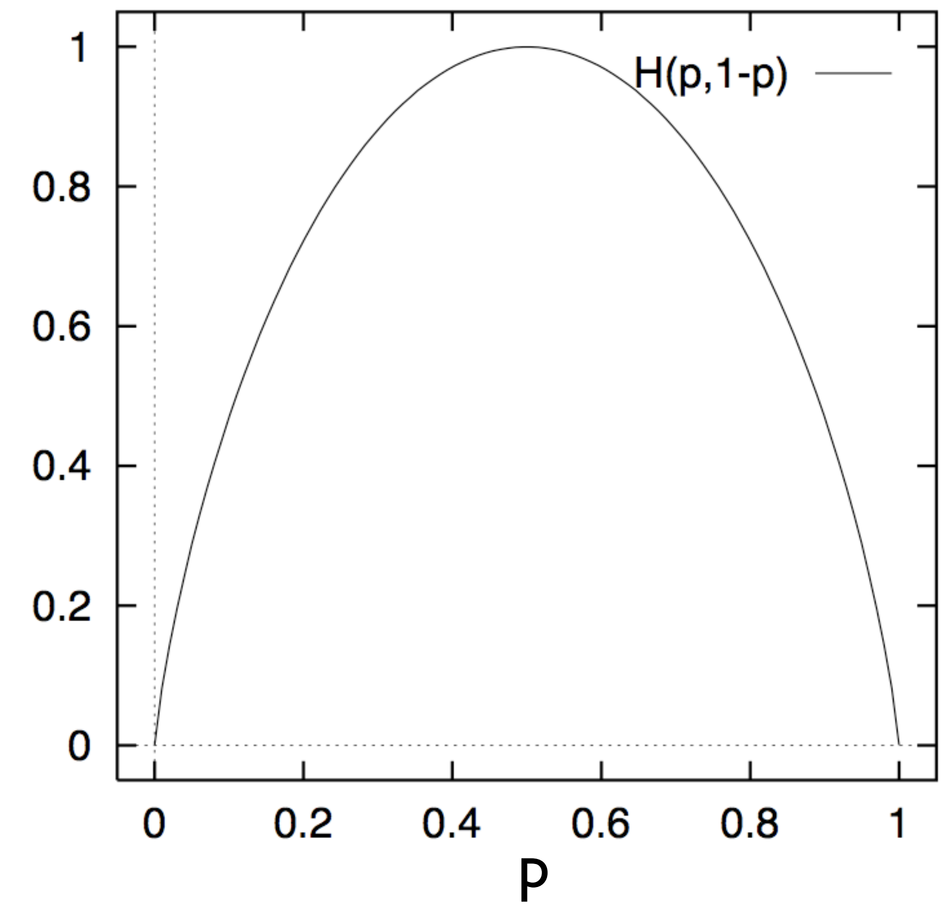
Example: Binary random variable X (Biased Coin)

$$\mathcal{X} = \{0, 1\} \qquad \Pr(1) = p \;\&\; \Pr(0) = 1 - p$$

$H(X)$?

Binary entropy function:

$$H(p) = -p \log_2 p - (1-p) \log_2 (1-p)$$



Fair coin: $p = \frac{1}{2}$

$$H(p) = 1 \text{ bit}$$

Completely biased coin: $p = 0 \ (\text{or } 1)$

$$H(p) = 0 \text{ bits}$$

Recall: $0 \cdot \log 0 = 0$

Information ...

Example: Independent, Identically Distributed (IID) Process over four events

$$\mathcal{X} = \{a, b, c, d\} \qquad \Pr(X) = (\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{8})$$

Entropy: $H(X) = \frac{7}{4}$ bits

Number of questions to identify the event?

x = a? (must always ask at least one question)

x = b? (this is necessary only half the time)

x = c? (only get this far a quarter of the time)

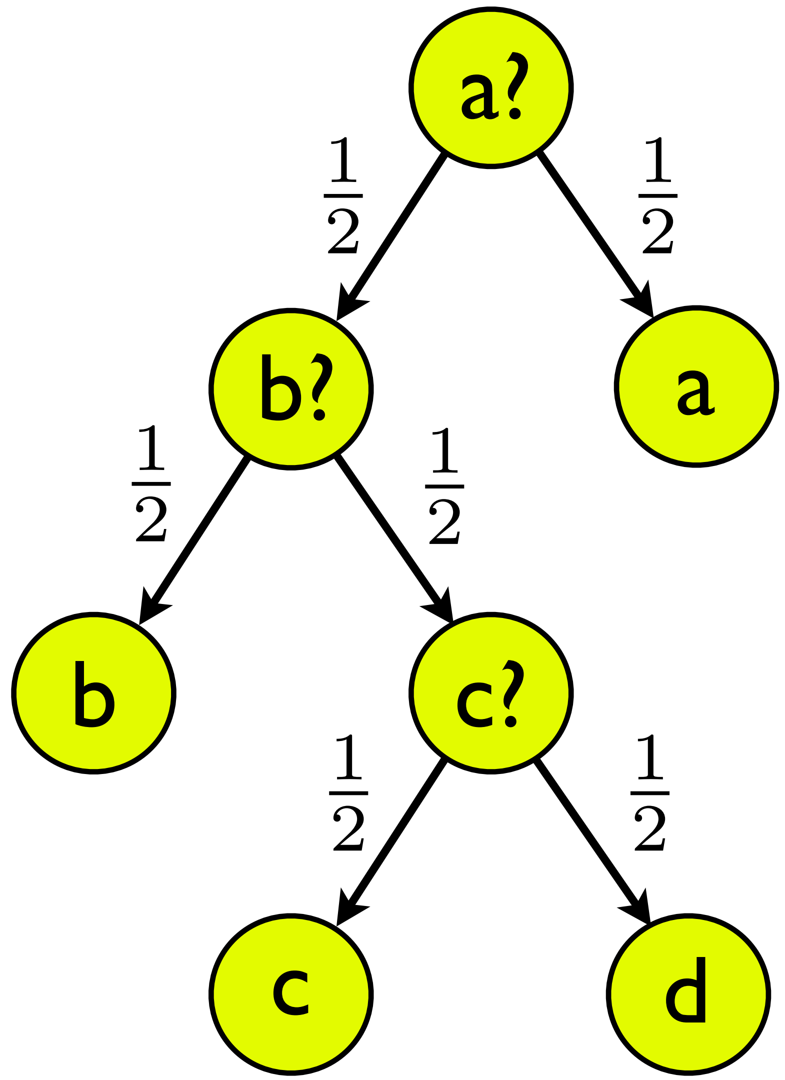
Average number: $1 \cdot 1 + 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = 1.75$ questions

Interpretation? Optimal way to ask questions.

Information ...

Example: IID Process over four events ...

Average number: $1 \cdot 1 + 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = 1.75$ questions



$$\Pr(X) = (\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{8})$$

Information ...

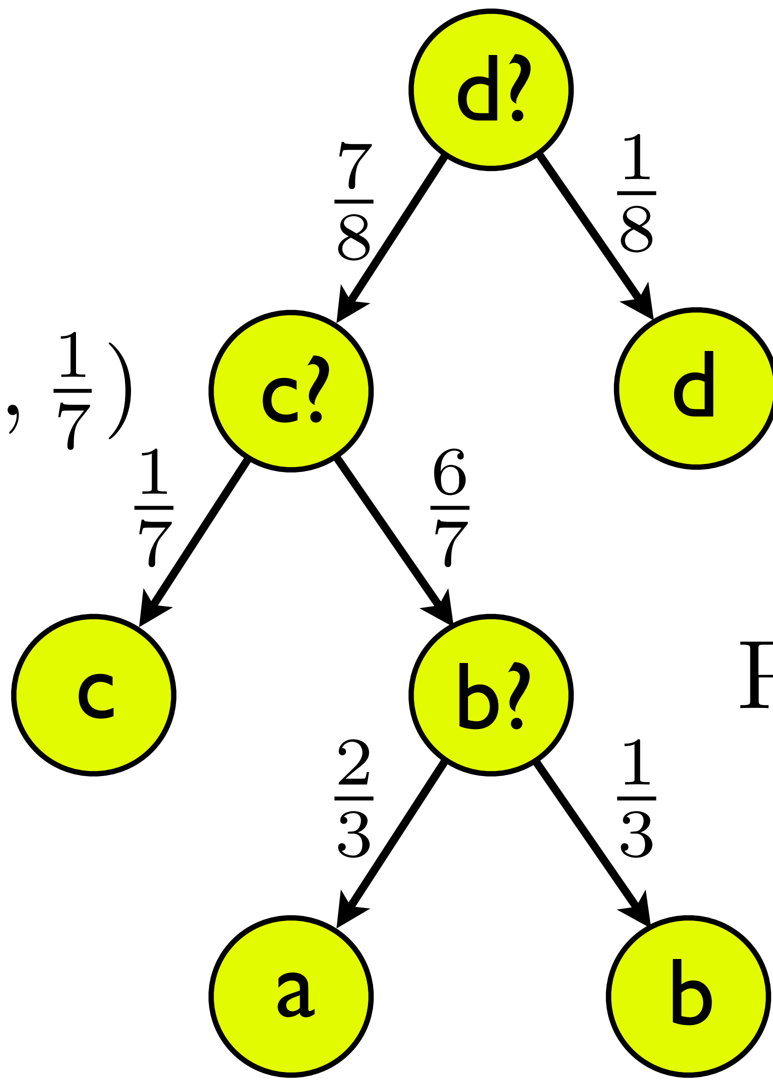
Example: IID Process over four events ...

Query in a different order:

Average number: $1 \cdot 1 + 1 \cdot \frac{7}{8} + 1 \cdot \frac{6}{7} \approx 2.7$ questions

$$\Pr(X) = (\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{8})$$

$$\Pr(a, b, c) = (\tfrac{4}{7}, \tfrac{2}{7}, \tfrac{1}{7})$$



$$\Pr(a, b) = (\tfrac{2}{3}, \tfrac{1}{3})$$

Information ...

Example: IID Process over four events

Entropy: $H(X) = \frac{7}{4}$ bits

At each stage, ask questions that are most informative.

Choose partitions of event space that give "most random" measurements.

Theorem:
Entropy gives the smallest number of questions to identify an event, on average.

Information ...

Interpretations of Shannon Entropy:

Observer's *degree of surprise* in outcome of a random variable

Uncertainty *in* random variable

Information required to *describe* random variable

A measure of *flatness* of a distribution

Information ...

Two random variables: $(X, Y) \sim p(x, y)$

Joint Entropy: Average uncertainty in X and Y occurring

$$H(X,Y) = -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x,y) \log_2 p(x,y)$$

Independent:

$$X \perp Y \Rightarrow H(X,Y) = H(X) + H(Y)$$

Conditional Entropy: Average uncertainty in X, knowing Y

$$H(X|Y) = -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x,y) \log_2 p(x|y)$$

$$H(X|Y) = H(X,Y) - H(Y)$$

Not symmetric: $H(X|Y) \neq H(Y|X)$

Common Information Between Two Random Variables:

$$X \sim p(x) \;\&\; Y \sim p(y)$$

$$(X, Y) \sim p(x, y)$$

Mutual Information:

$$I(X;Y) = \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} p(x,y) \log_2 \frac{p(x,y)}{p(x)p(y)}$$

Information ...

Mutual Information ...

Properties:

(1) $I(X;Y) \geq 0$

(2) $I(X;Y) = I(Y;X)$

(3) $I(X;Y) = H(X) - H(X|Y)$

(4) $I(X;Y) = H(X) + H(Y) - H(X,Y)$

(5) $I(X;X) = H(X)$

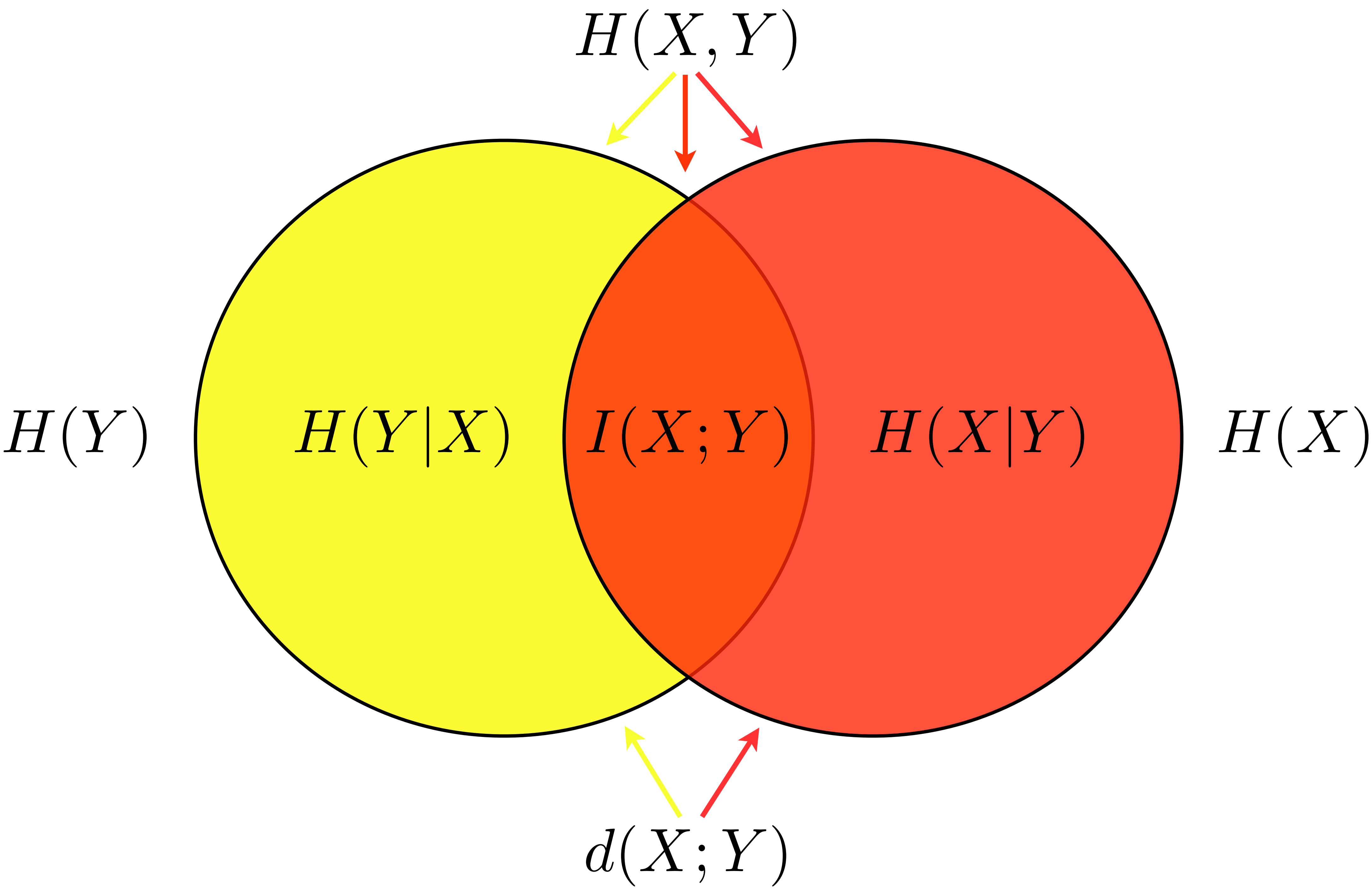
(6) $X \perp Y \Rightarrow I(X;Y) = 0$

Interpretations:

Information one variable has about another
Information shared between two variables
Measure of dependence between two variables

Information ...

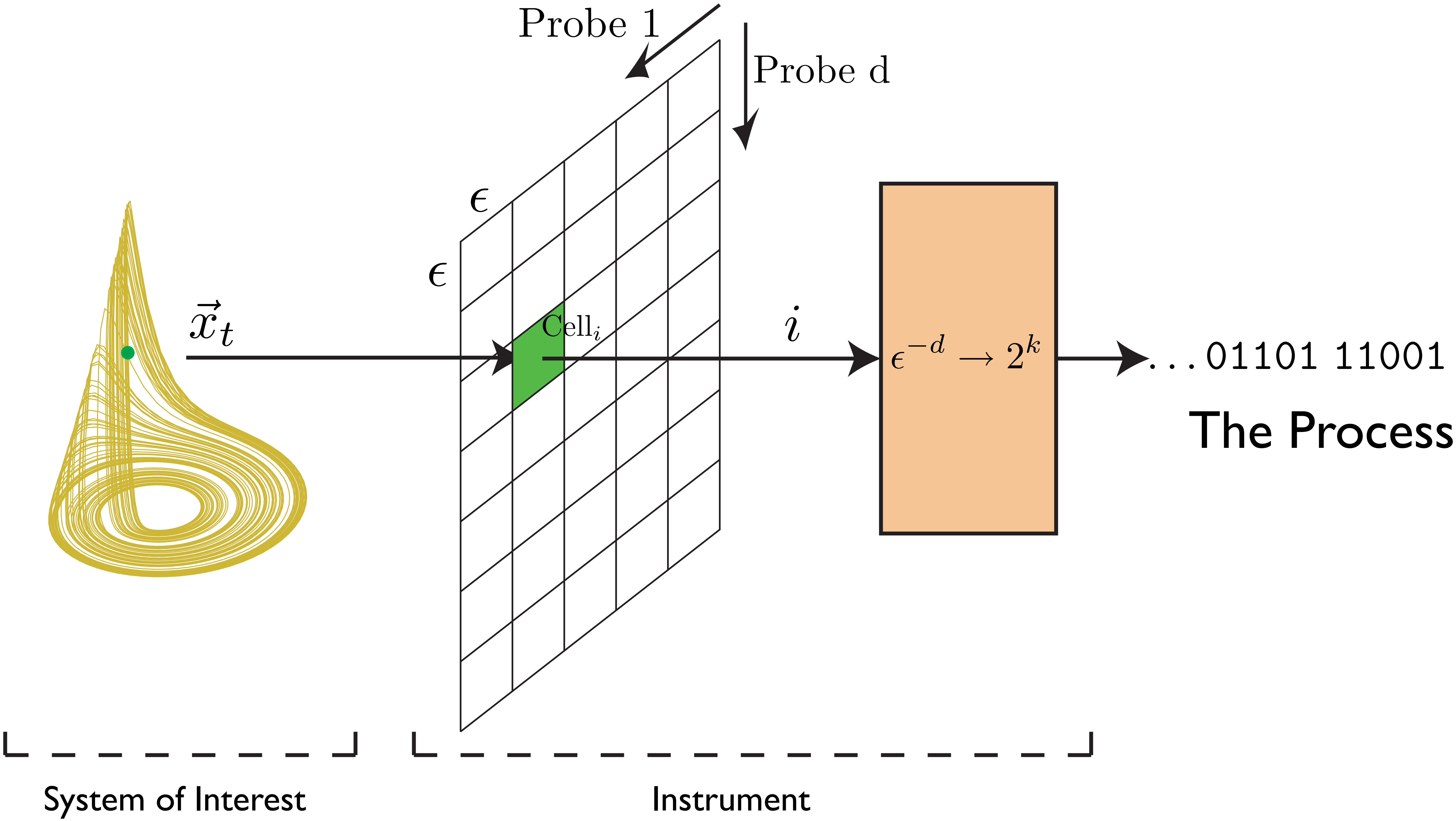
Event Space Relationships of Information Quantifiers:



Why information?

1. Accounts for any type of co-relation
   - Statistical correlation ~ linear only
   - Information measures nonlinear correlation
2. Broadly applicable:
   - Many systems don't have "energy", physical modeling precluded
   - Information defined: social, biological, engineering, ... systems
3. Comparable units across different systems:
   - Correlation: Meters v. volts v. dollars v. ergs v. ...
   - Information: bits.
4. Probability theory ~ Statistics ~ Information
5. Complex systems:
   - Emergent patterns!
   - We don't know these ahead of time

Processes and Their Models



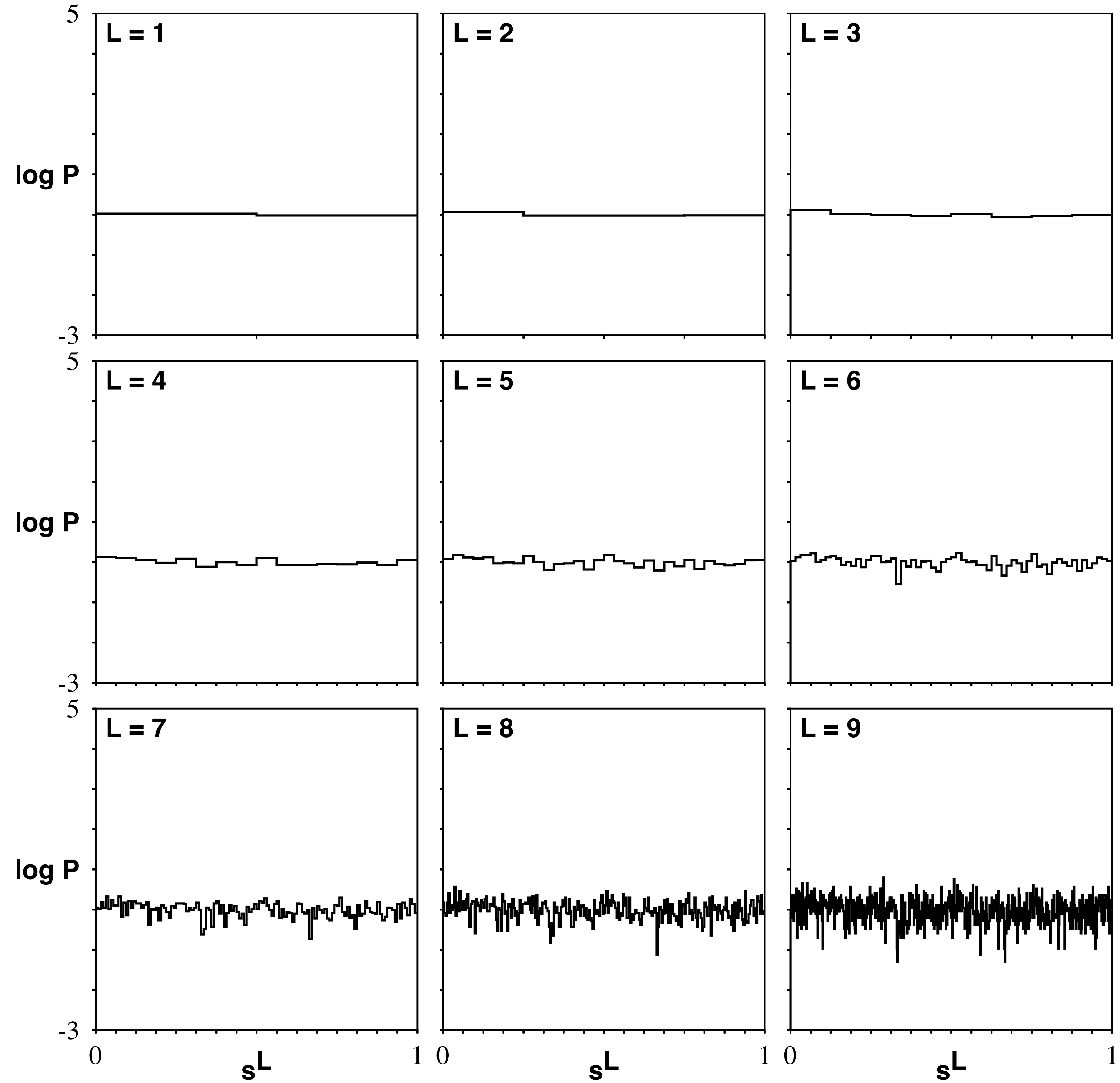
Measurement Channel

Processes and Their Models ...

Models of Stochastic Processes ...

Fair Coin ...

Sequence Distribution: $\Pr(v^L) = 2^{-L}$



Word as binary fraction:

$$s^L = s_1 s_2 \ldots s_L$$

$$\text{“}s^L\text{”} = \sum_{i=1}^{L} \frac{s_i}{2^i}$$

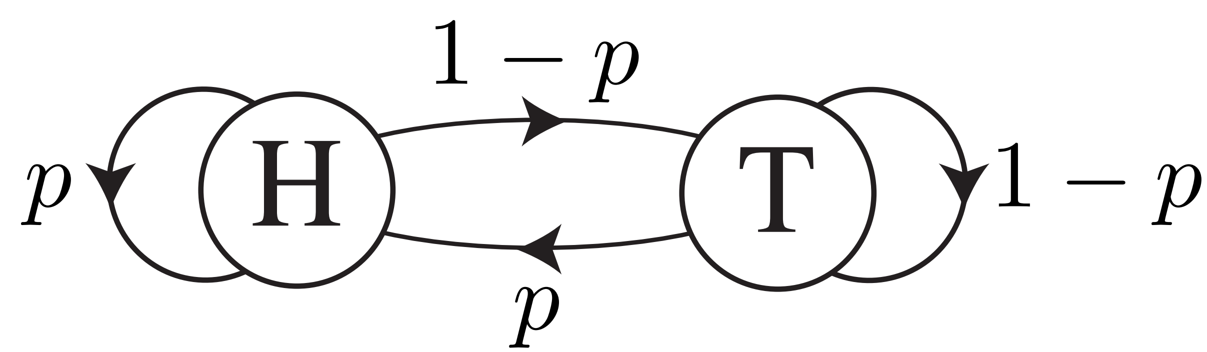
$$s^L \in [0, 1]$$

Processes and Their Models ...

Models of Stochastic Processes ...

Biased Coin: $\mathcal{A} = \{H, T\}$

$$T = \begin{pmatrix} p & 1-p \\ p & 1-p \end{pmatrix}$$



$$\Pr(H) = p$$

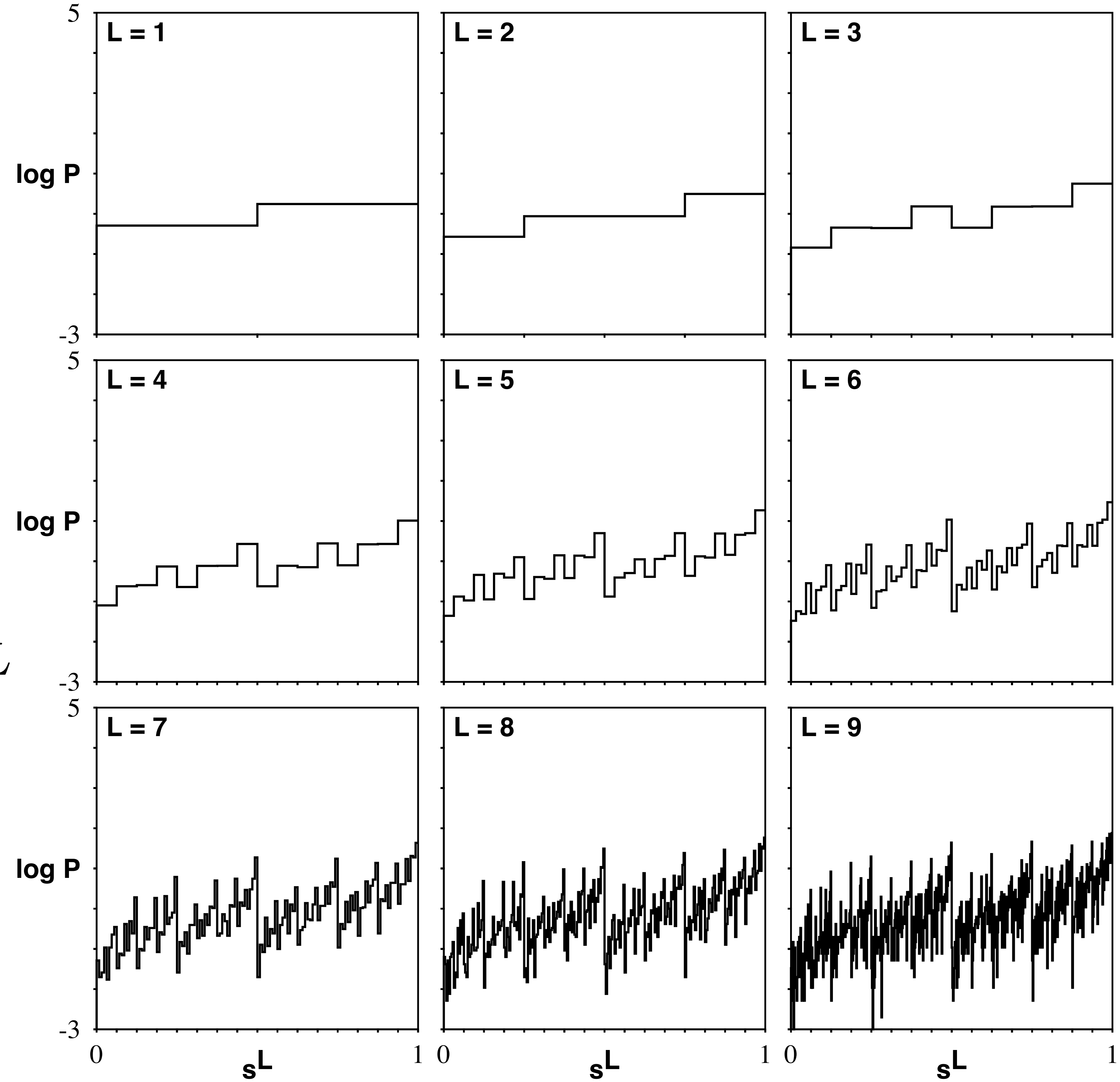
$$\Pr(T) = 1 - p$$

$$\pi = \Pr(p, 1-p)$$

Processes and Their Models ...

Models of Stochastic Processes ...

Biased Coin ...



Sequence Distribution:

$$\Pr(s^L) = p^n (1-p)^{L-n},$$

$$n = \text{Number } H\text{s in } s^L$$

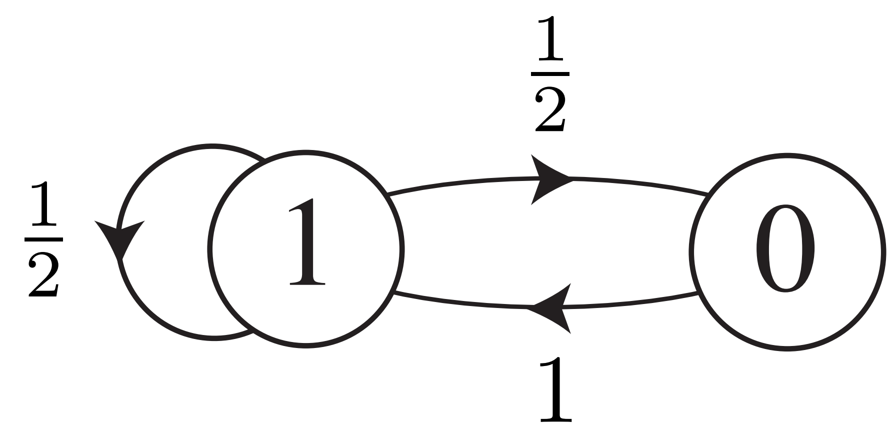
Processes and Their Models ...

Models of Stochastic Processes ...

Golden Mean Process = "No consecutive 0s"

Markov chain over 1-Blocks: $\mathcal{A} = \{0, 1\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$



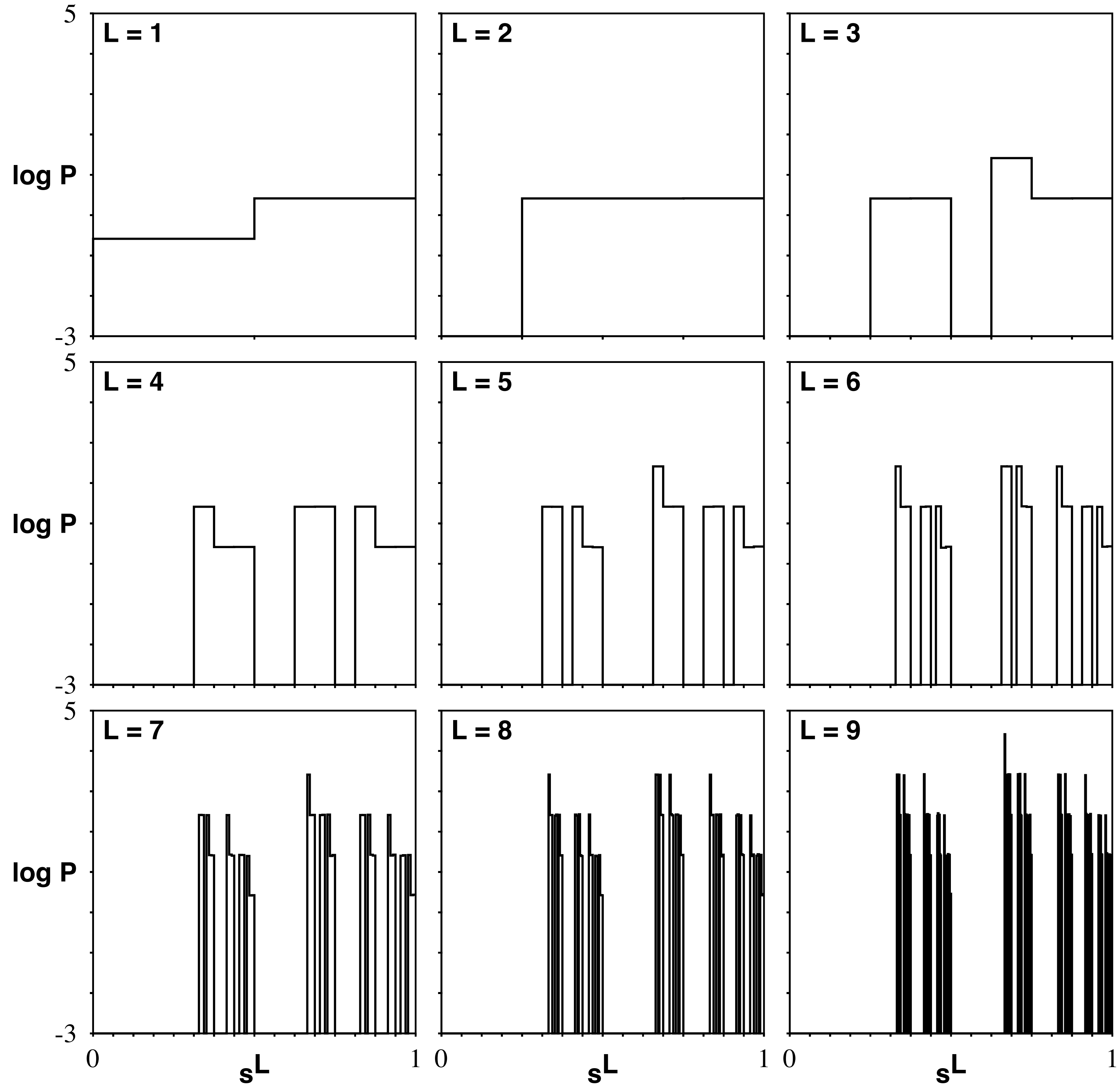
$$\pi = \Pr(V = 1, V = 0)$$
$$= \left(\tfrac{2}{3}, \tfrac{1}{3}\right)$$

As an order-1 Markov chain.
A minimal-order model of the GM Process.

Processes and Their Models ...

Models of Stochastic Processes ...

Golden Mean:



Models of Stochastic Processes ...

Two Lessons:

Structure in the behavior: $\mathrm{supp}\ \Pr(s^L)$

Structure in the distribution of behaviors: $\Pr(s^L)$

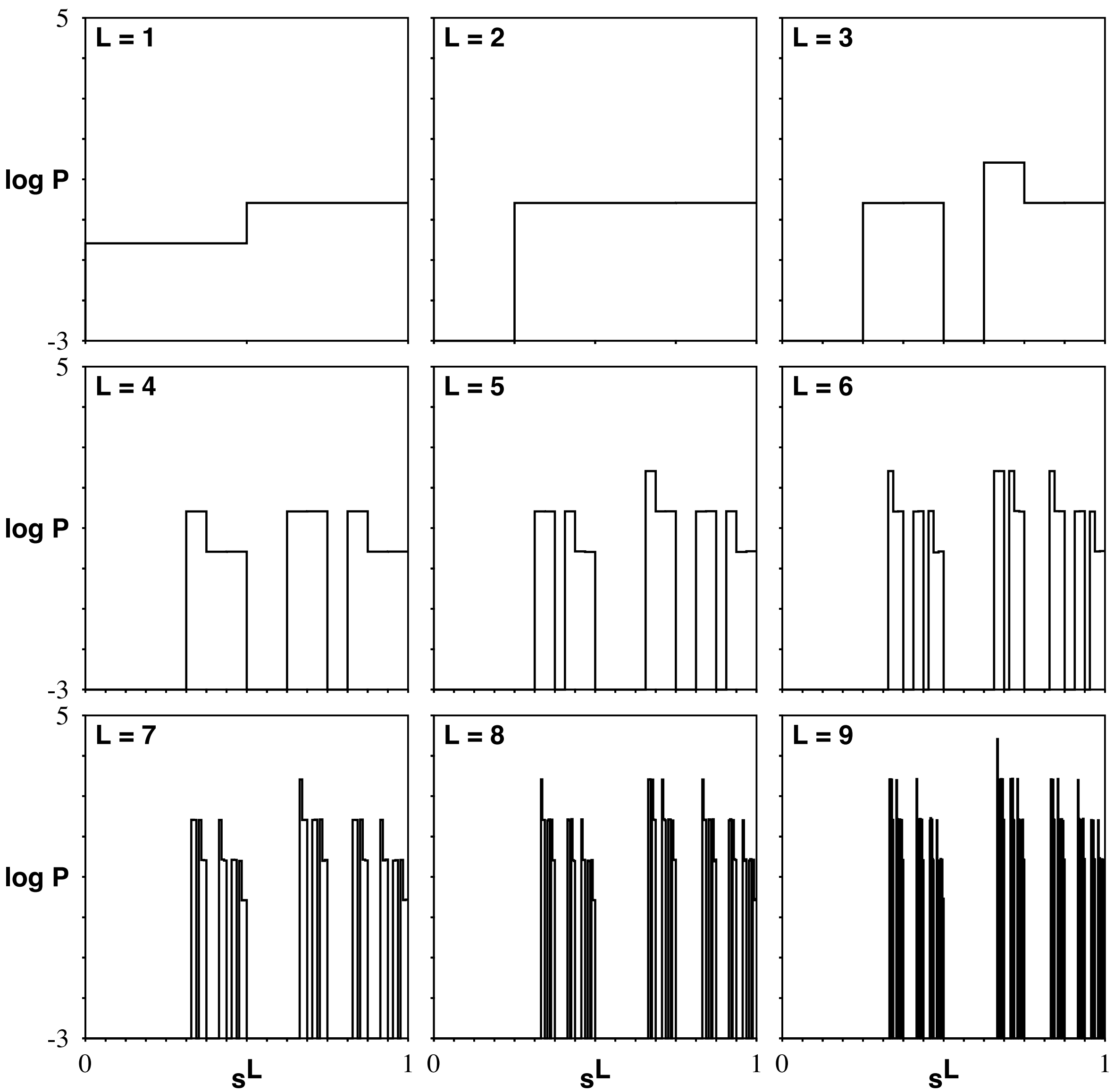
Processes and Their Models ...

Models of Stochastic Processes ...

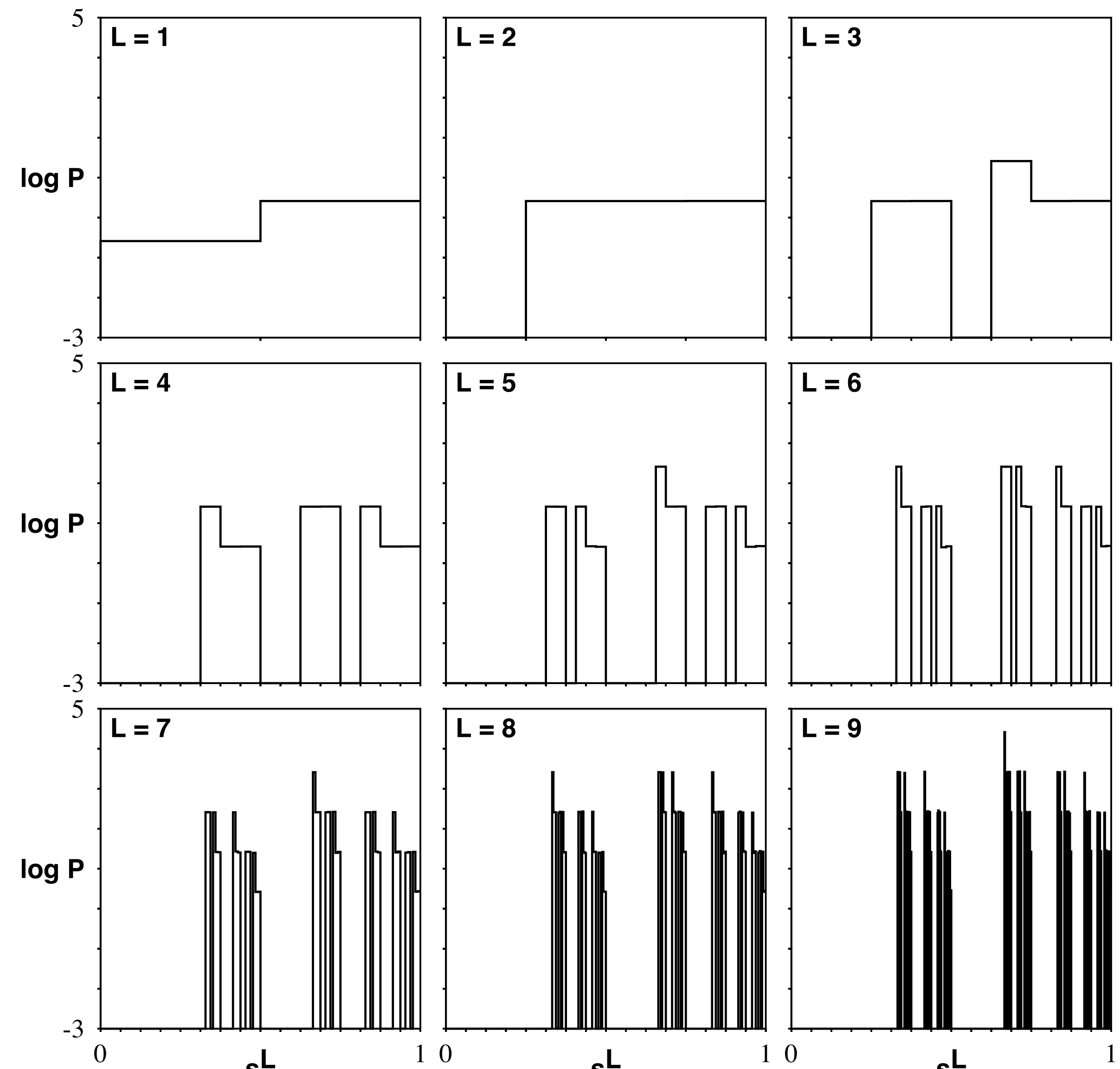
Golden Mean Process ... Sequence distributions:

Internal state sequences

(A = 1; B = 0)



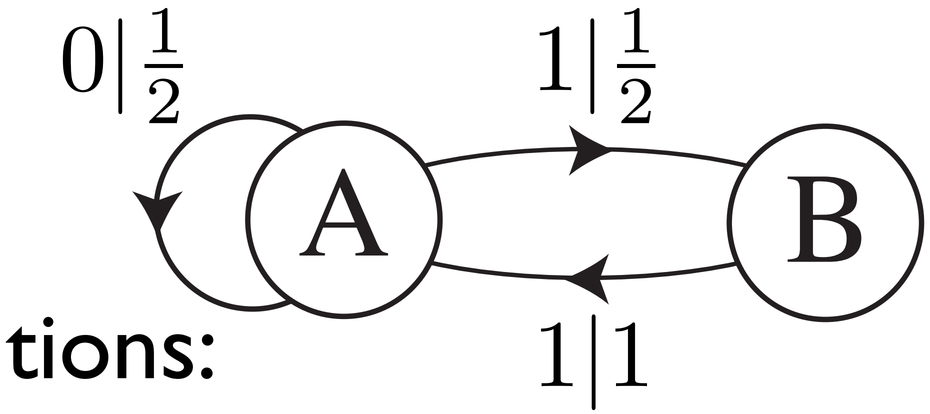
Observed sequences



Same!

Processes and Their Models ...

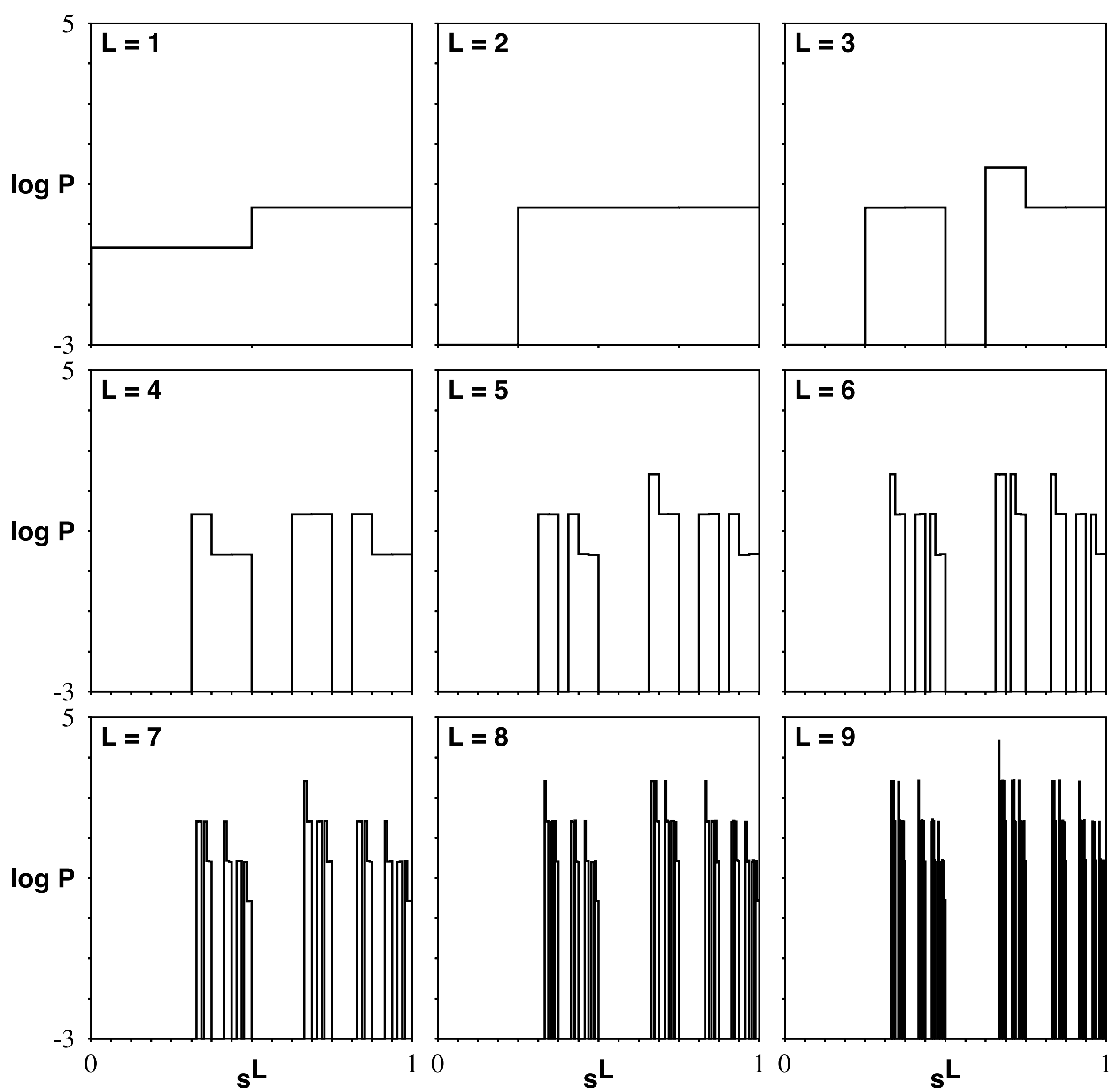
Models of Stochastic Processes ...



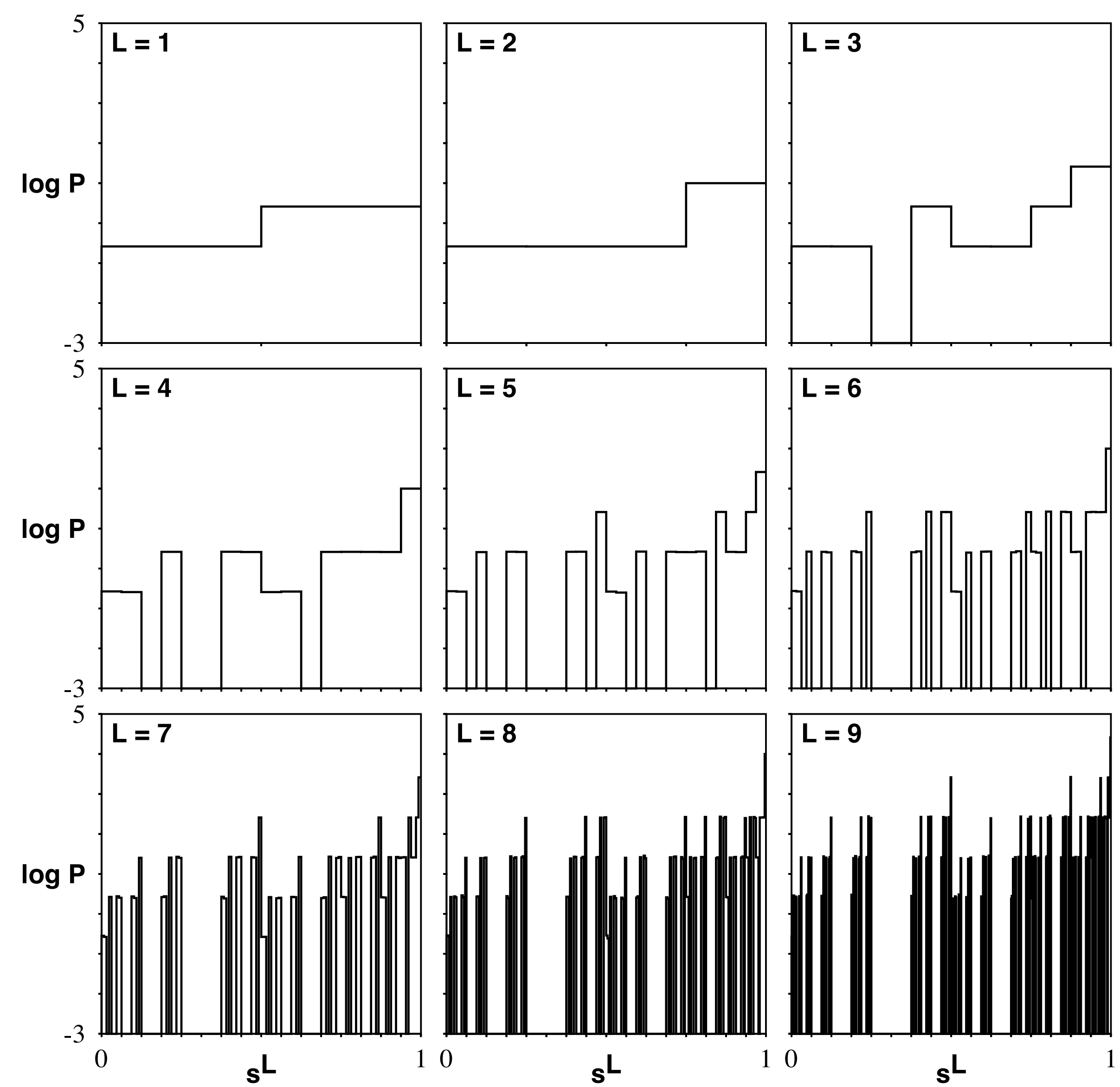
Even Process ... Sequence distributions:

Internal states (= GMP)

(A = 1; B = 0)



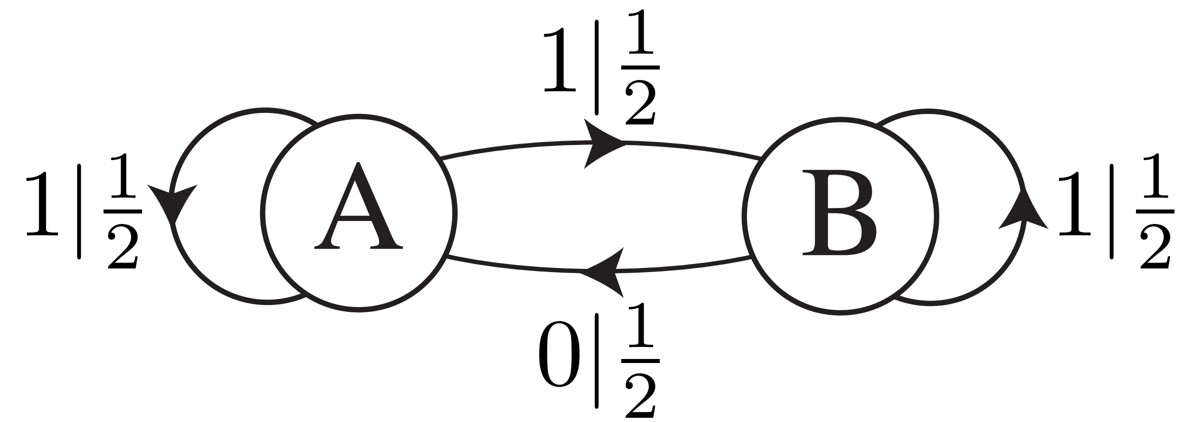
Observed sequences



Rather different!

Processes and Their Models ...

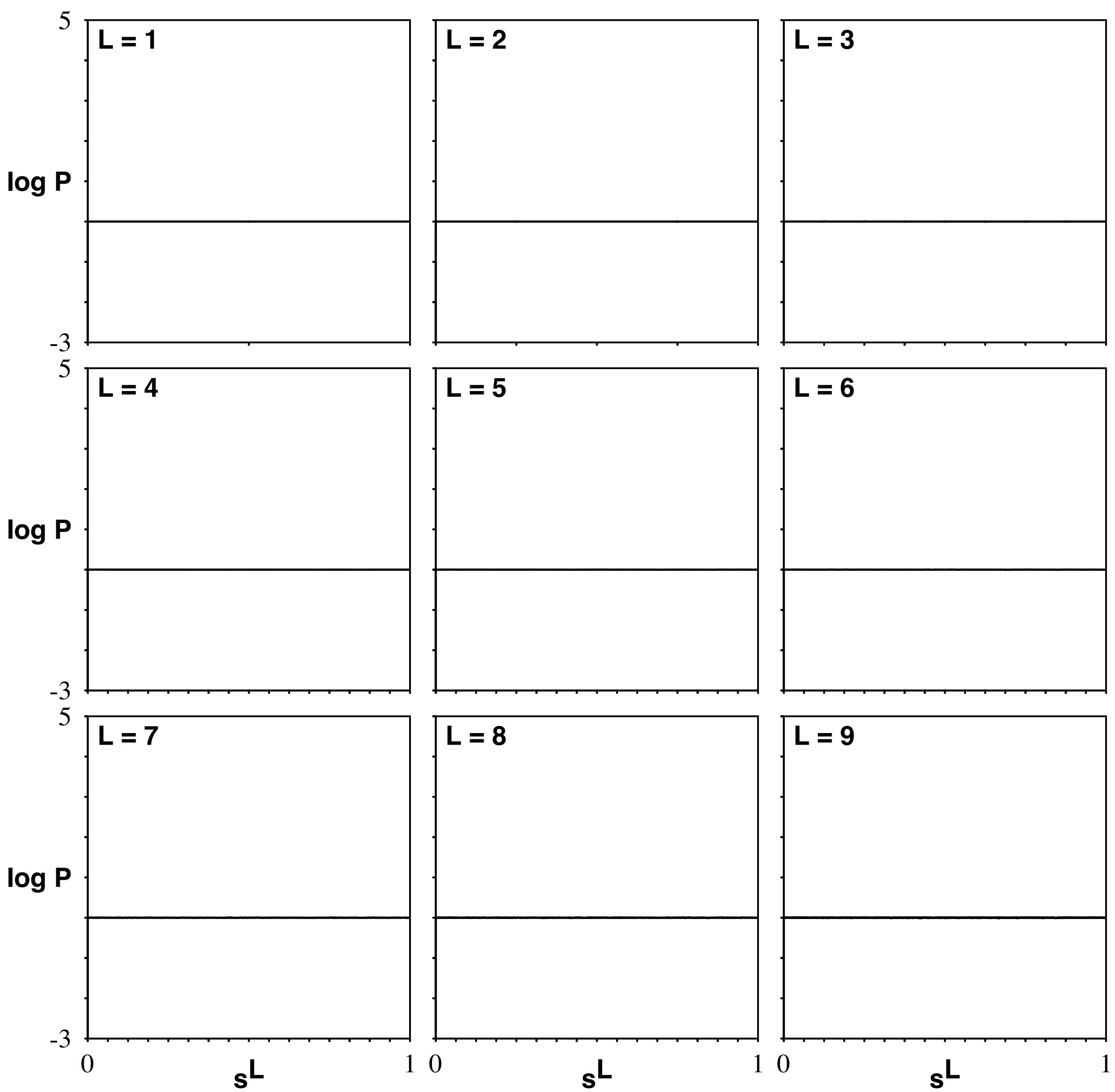
Models of Stochastic Processes ...



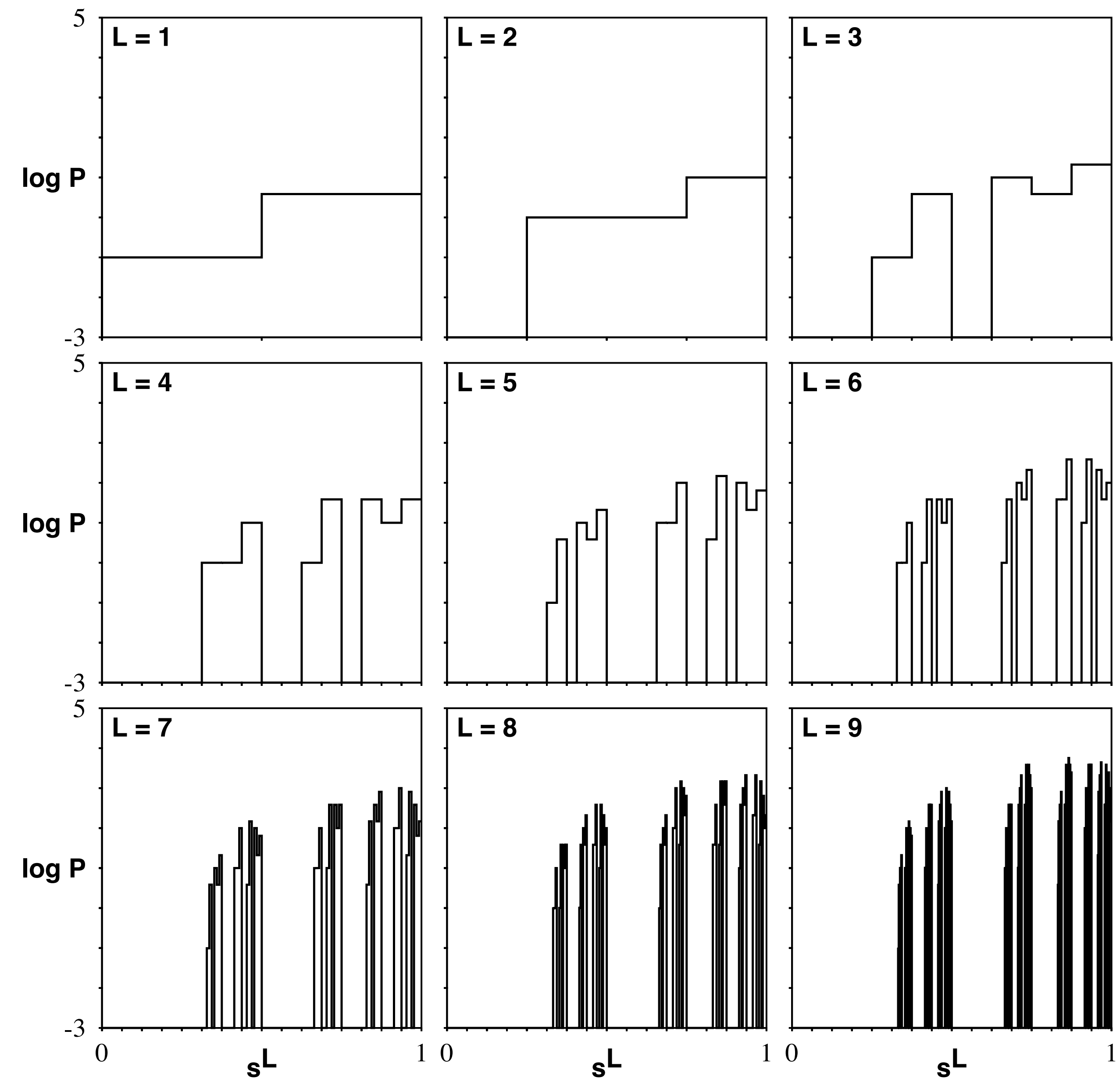
Simple Nonunifilar Source ...

Internal states (= Fair coin)

(A = 1; B = 0)



Observed sequences



Information in Processes

Entropy Growth for Stationary Stochastic Processes: $\Pr(\overleftrightarrow{S})$

Block Entropy:

$$H(L) = H(\Pr(s^L)) = -\sum_{s^L \in \mathcal{A}} \Pr(s^L) \log_2 \Pr(s^L)$$

Monotonic increasing: $H(L) \geq H(L-1)$

Adding a random variable cannot decrease entropy:

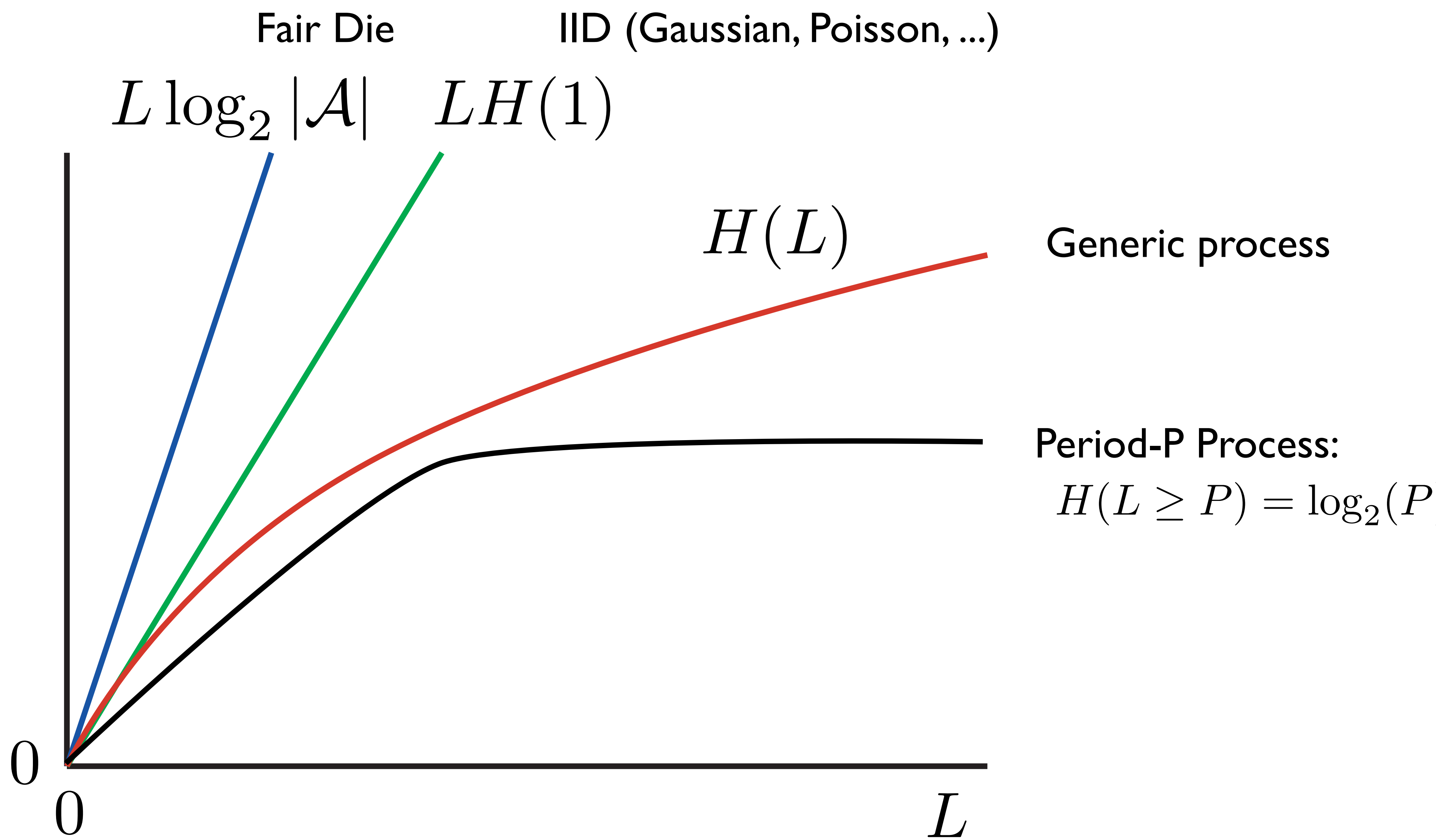
$$H(S_1, S_2, \ldots, S_L) \leq H(S_1, S_2, \ldots, S_L, S_{L+1})$$

No measurements, no information: $H(0) = 0$

Information in Processes ...

Entropy Growth for Stationary Stochastic Processes ...

Block Entropy ...

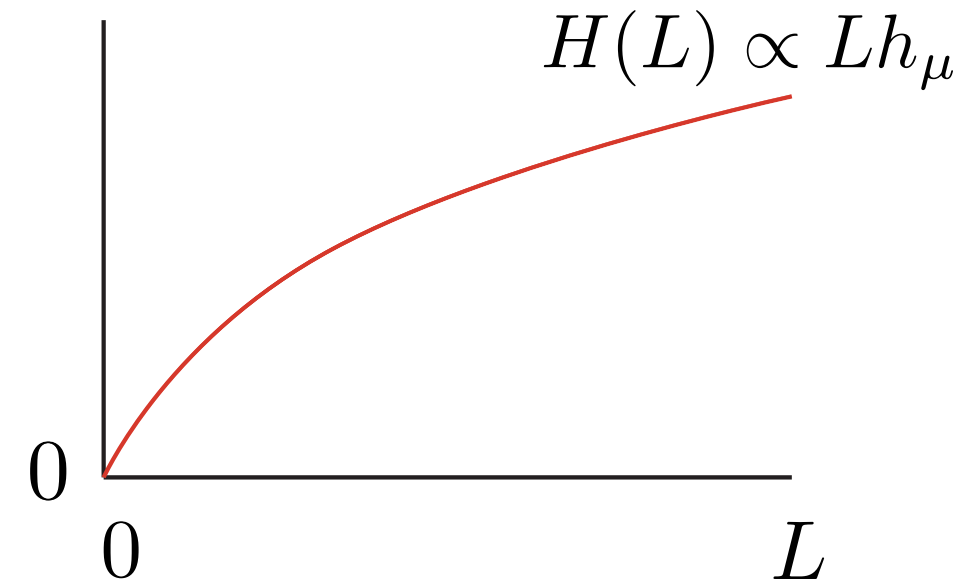


Entropy Rates for Stationary Stochastic Processes:

Entropy per symbol is given by the **Source Entropy Rate**:

$$h_\mu = \lim_{L \to \infty} \frac{H(L)}{L}$$

(When limits exists.)



Interpretations:

- Asymptotic growth rate of entropy
- Irreducible randomness of process
- Average description length (per symbol) of process

Information in Processes ...

Entropy Convergence:

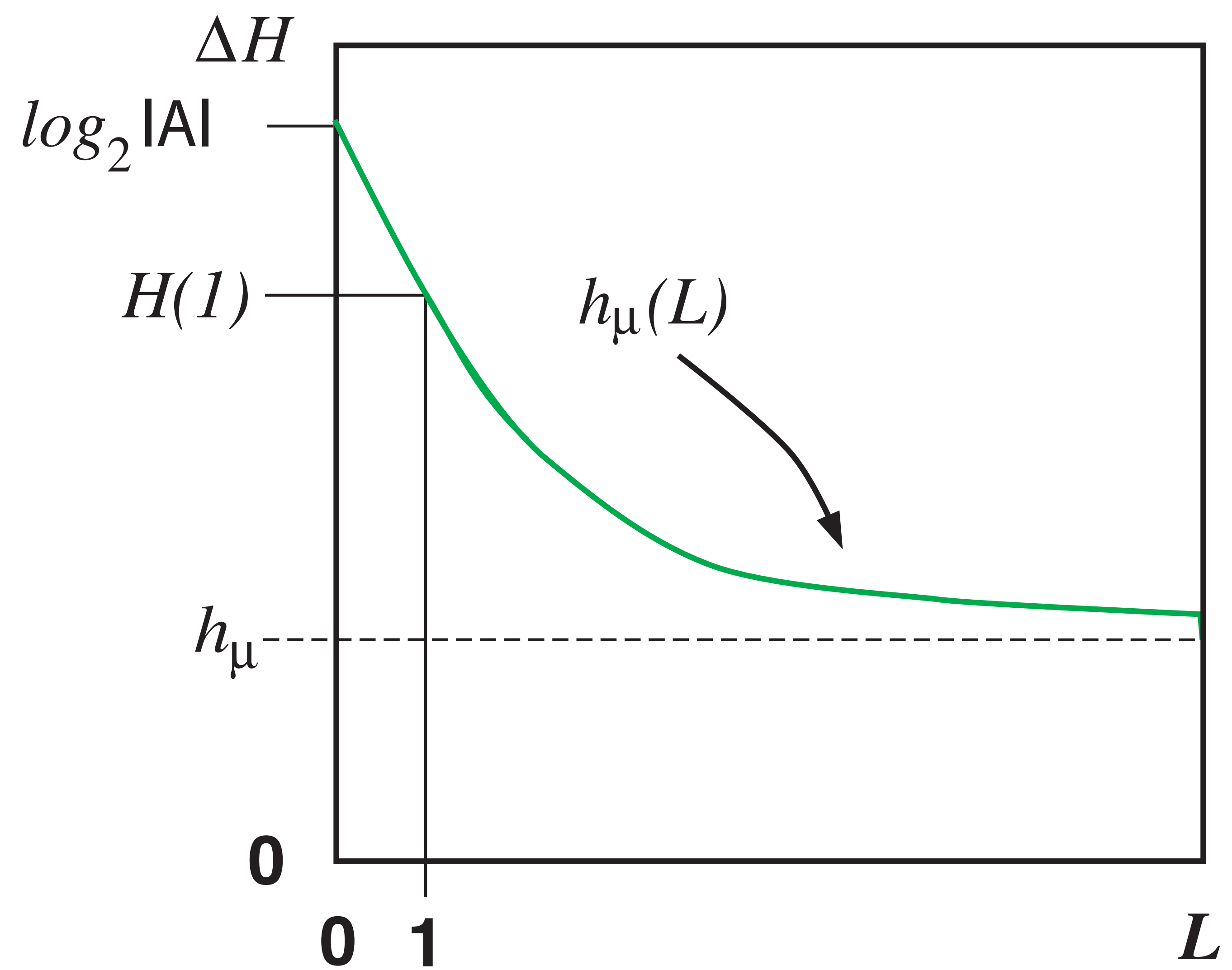
Length-L entropy rate estimate:

$$h_\mu(L) = H(L) - H(L-1)$$

$$h_\mu(L) = \Delta H(L)$$

Monotonic decreasing:

$$h_\mu(L) \leq h_\mu(L-1)$$



Process appears less random
as account for longer correlations

Memory in Processes

Information in Processes ...

Motivation:

Previous: Measures of randomness of information source

Block entropy $H(L)$

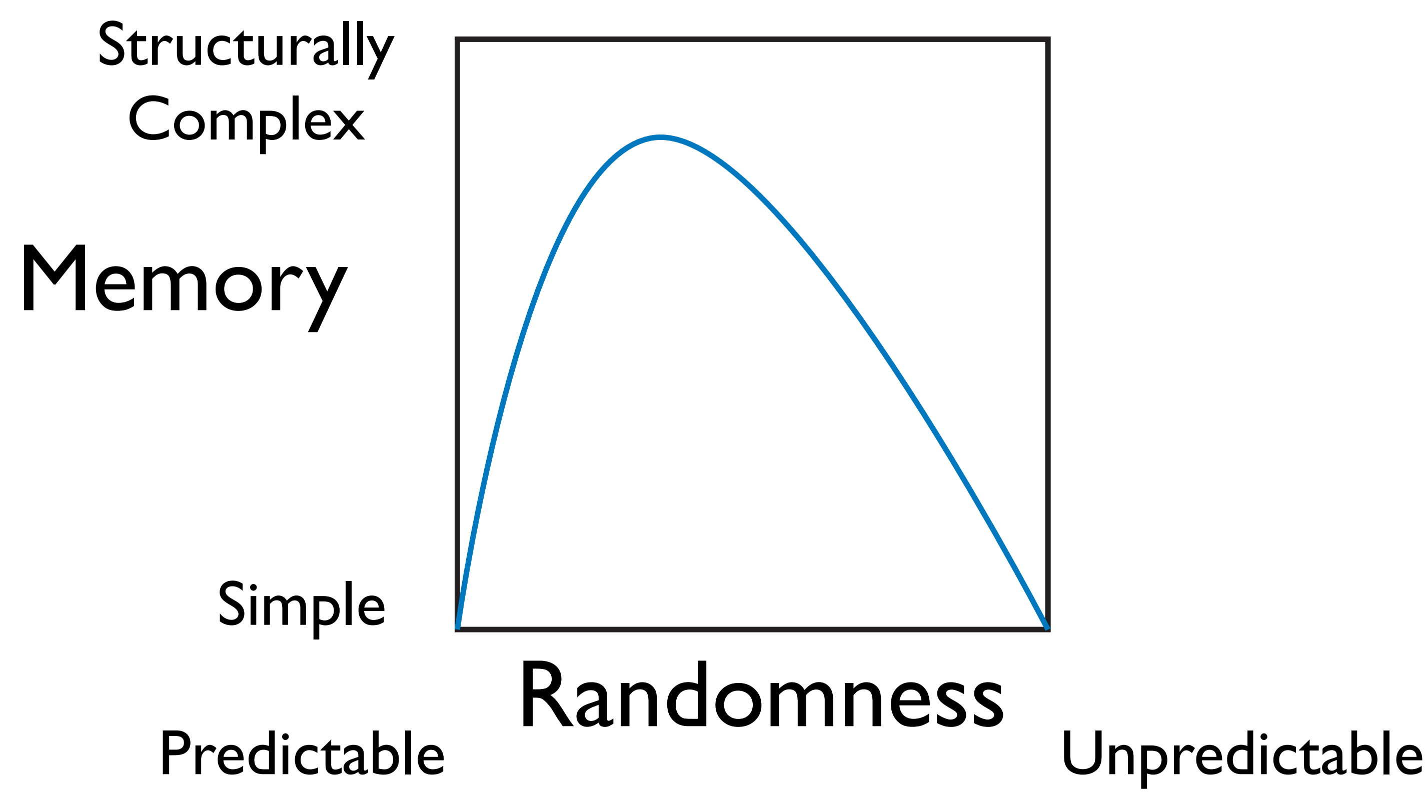
Entropy rate h_μ

Current target point:

Measures of memory & information storage

Big Picture:

Complementary.



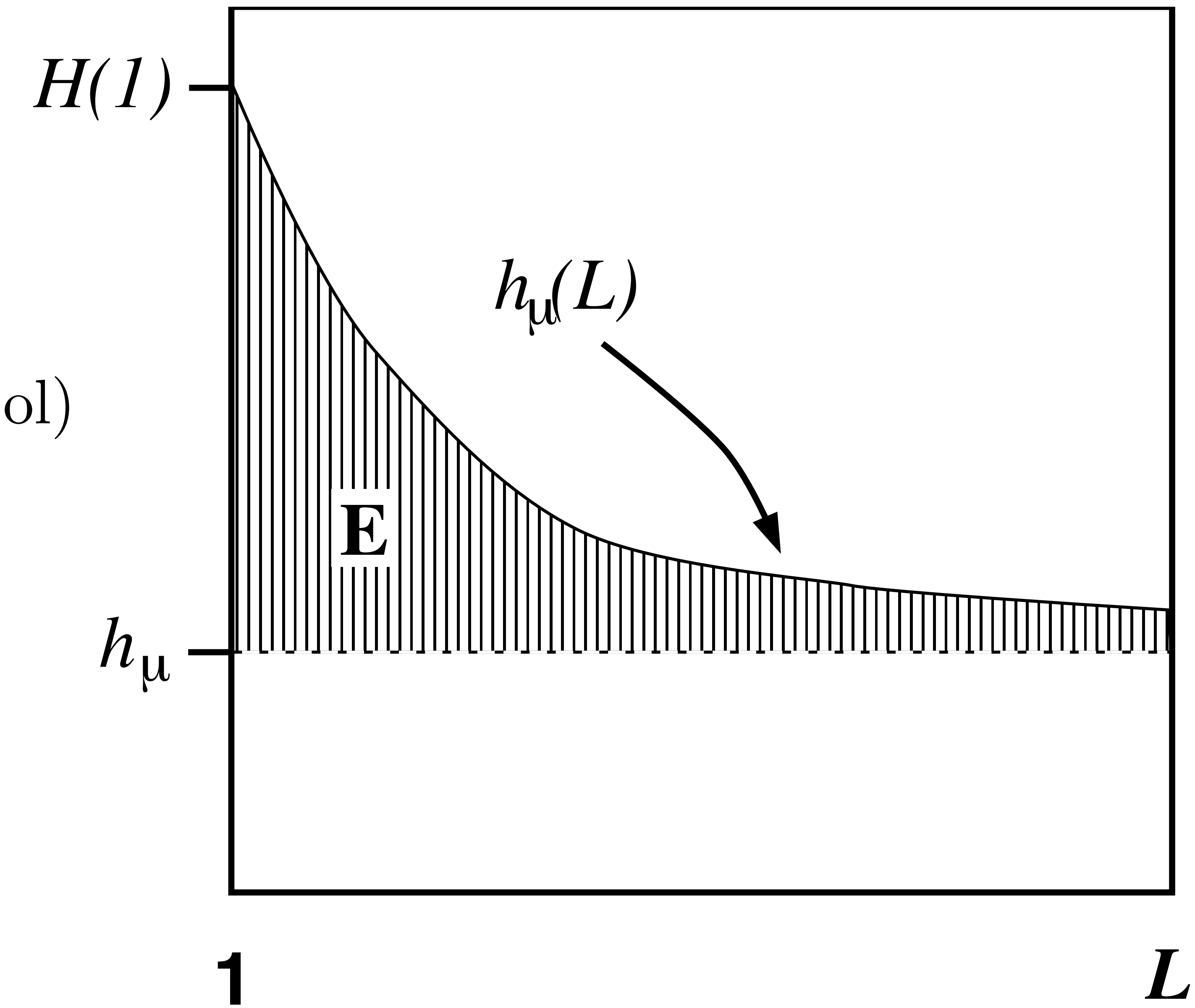
Memory in Processes ...

Excess Entropy:

As entropy convergence:

$$\mathbf{E} = \sum_{L=1}^{\infty} [h_\mu(L) - h_\mu]$$

$(\Delta L = 1 \text{ symbol})$



Properties:

(1) Units: $\mathbf{E} = [\text{bits}]$

(2) Positive: $\mathbf{E} \geq 0$

(3) Controls convergence to actual randomness.

(4) Slow convergence ⇔ Correlations at longer words.

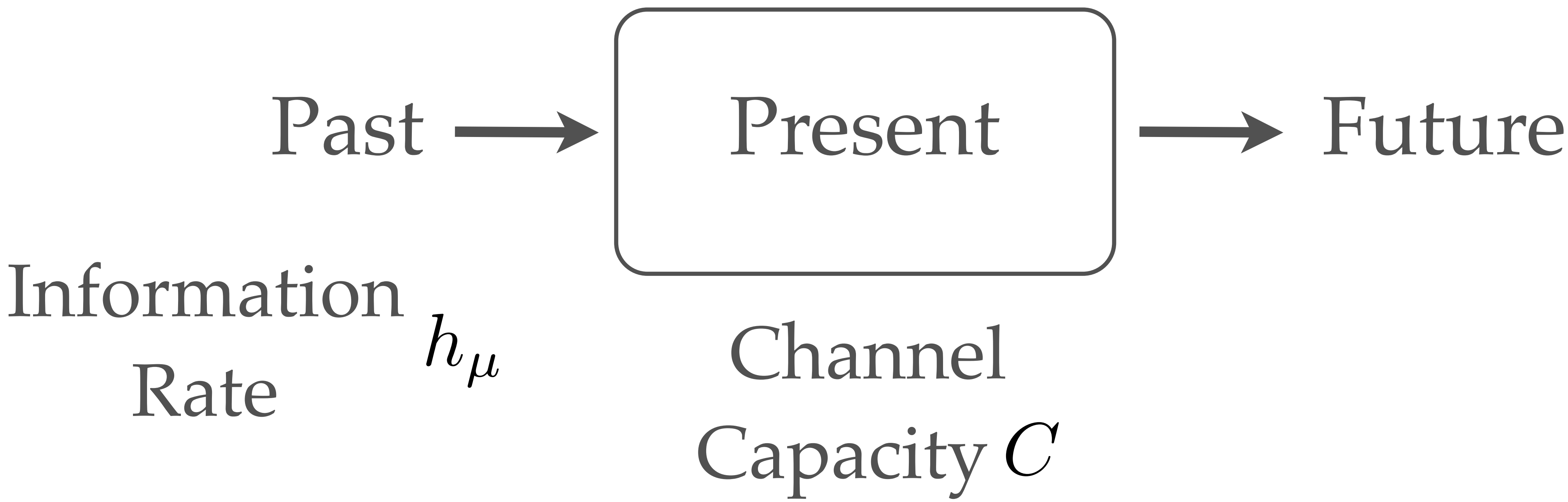
(5) Complementary to entropy rate.

Memory in Processes ...

Excess Entropy ...

Mutual information between past and future: Process as channel

Process $\Pr(\overleftarrow{X}, \overrightarrow{X})$ communicates past $\overleftarrow{X}$ to future $\overrightarrow{X}$:

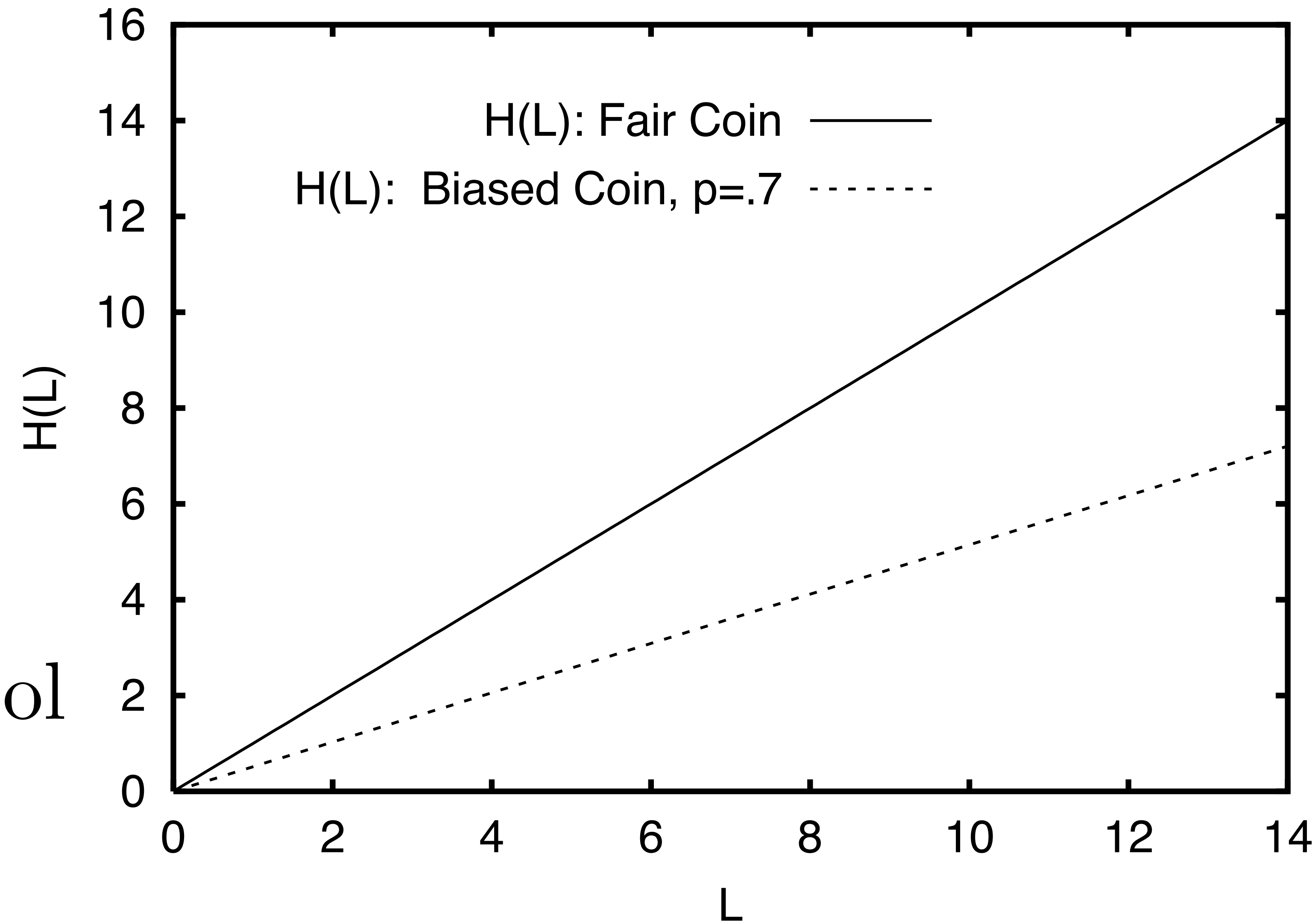


Excess Entropy as Channel Utilization:

$$\mathbf{E} = I[\overleftarrow{X}; \overrightarrow{X}]$$

Memory in Processes ...

Examples of Excess Entropy:



Fair Coin:

$$h_\mu = 1 \text{ bit per symbol}$$

$$\mathbf{E} = 0 \text{ bits}$$

Biased Coin:

$$h_\mu = H(p) \text{ bits per symbol}$$

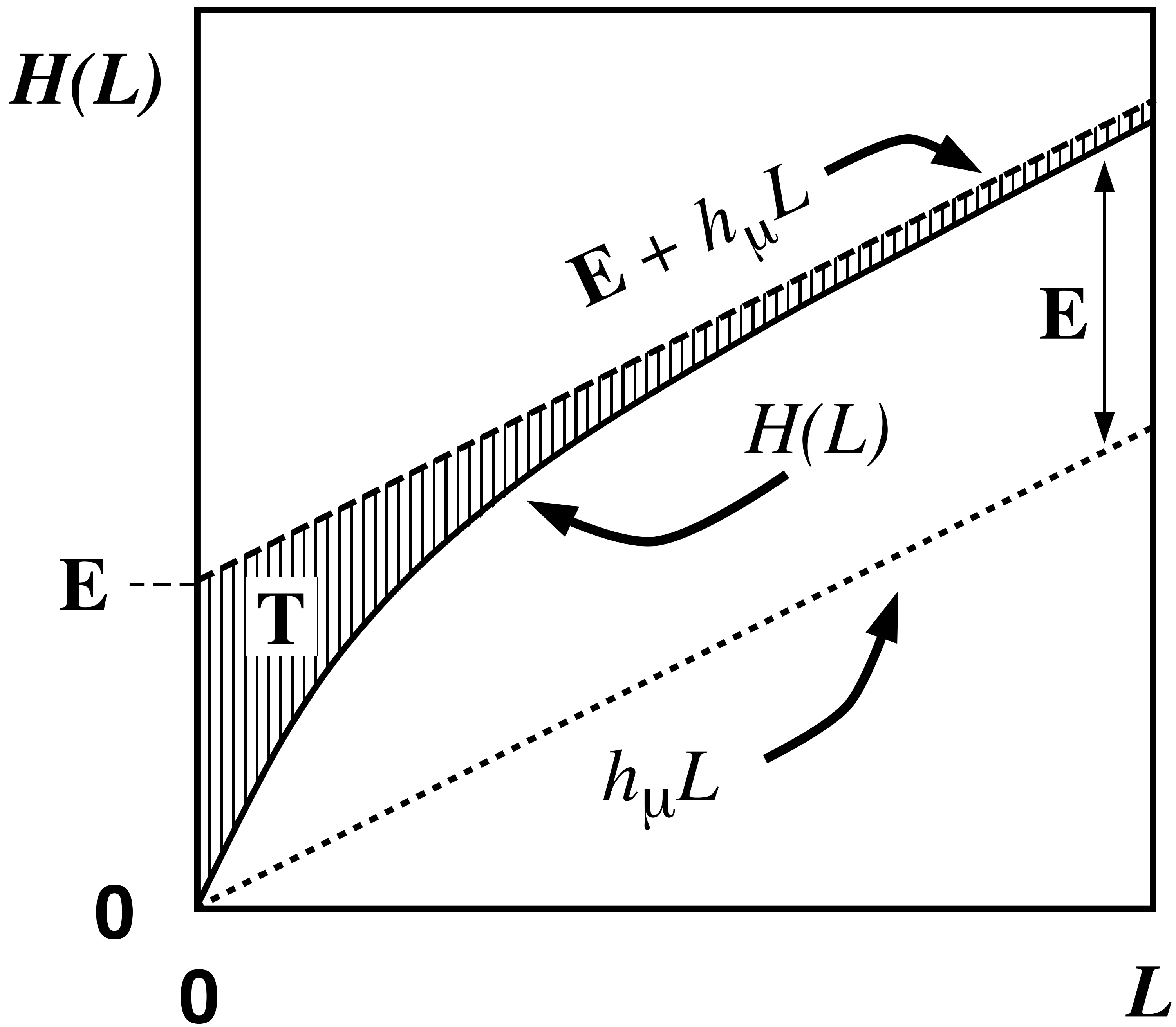
$$\mathbf{E} = 0 \text{ bits}$$

Any IID Process:

$$h_\mu = H(X) \text{ bits per symbol}$$

$$\mathbf{E} = 0 \text{ bits}$$

Information-Entropy Roadmap for a Stochastic Process:



Memory in Processes ...

What is information?

Depends on the question!

Uncertainty, surprise, randomness, ...
Compressibility.
Transmission rate.
Memory, apparent stored information, ...
Synchronization.
...

Algorithmic Basis of Information

Kolmogorov-Chaitin Complexity versus Shannon Information

KC Complexity versus Shannon Information

Consider average KC Complexity of source:

$$K(\ell) \equiv \langle K(x_{0:\ell}) \rangle_{\text{realizations}}$$

Recall Block Entropy:

$$H(\ell) \equiv H[\Pr(X_{0:\ell})]$$

Their growth rates equal the Shannon entropy rate:

$$h_\mu = \lim_{\ell \to \infty} \frac{H(\ell)}{\ell} = \lim_{\ell \to \infty} \frac{K(\ell)}{\ell}$$

KC Complexity of typical realizations from an information source grows proportional to the Shannon entropy rate [Brudno 1978].

KC Complexity versus Shannon Information

Again, KC Complexity is a measure of randomness, unpredictability, surprise, ...

As well as being a measure of the *deterministic* computing resources requires to *exactly* reproduce a given finite string.

KC Complexity and entropy rate maximized by IID processes.

KC Complexity versus Statistical Complexity

KC Complexity Theory:

- Great mathematics.
- Uncomputable.
- Not quantitative: constants of proportionality unknown

Quantitative sciences use Information Theory instead.

Information in Complex Systems

Done:
- Algorithmic Basis of Probability
- Information Theory
- Information Measures

Next:
- Measuring Structure
- Intrinsic Computation
- Optimal Models
- Physics of Information

See online course:
http://csc.ucdavis.edu/~chaos/courses/ncaso/