Introduction to Nonlinear Dynamics

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Chaos

Complex behavior, arising in a deterministic nonlinear dynamic system, which exhibits two special properties:

• sensitive dependence on initial conditions
• characteristic structure…

Systems that exhibit chaos are ubiquitous; many of them are also simple, well-known, and “well-understood”

Where nonlinear dynamics turns up

• Flows (of fluids, heat, …)
  - Eddy in creek
  - Weather
  - Vortices around marine invertebrates
  - Air/fuel flow in combustion chambers

Where nonlinear dynamics turns up

• Driven nonlinear oscillators
  - Pendula
  - Hearts
  - Fireflies

  - and lots of other electronic, chemical, & biological systems
Where nonlinear dynamics turns up

- Classical mechanics
  - three-body problem
  - paired black holes
  - pulsar emission
- Protein folding
- Population biology
- And many, many other fields (including yours)

A useful graphical solution technique

- “cobweb” diagram
- aka return map
- aka correlation plot

Bifurcations

 Qualitative changes in the dynamics caused by changes in parameters:

- Heart: pathology
- Eddy in creek: water level
- Olfactory bulb: smell
- Brain: blood chemicals
- etc. etc.
Bifurcations in the logistic map

Note: in discrete time plots, it makes no sense to connect dots!!

Plots from Strogatz

• chaos
• veils/bands: places where chaotic attractor is dense (UPOs)
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• veils/bands: places where chaotic attractor is dense (UPOs)
• period-doubling cascade @ low R

Feigenbaum number

Universality!
Feigenbaum number and many other interesting chaotic/dynamical properties hold for any 1D map with a quadratic maximum.
Proof: renormalizations. See Strogatz §10.7

Don’t take this too far, though…

• chaos
• veils/bands: places where chaotic attractor is dense (UPOs)
• period-doubling cascade @ low R
• windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)
A bit more lore on periods and chaos

- Sarkovskii (1964)
  \[3, 5, 7, \ldots, 3x2, 5x2, \ldots, 3x2^3, 5x2^3, \ldots, 2^3, 2, 1\]
- Yorke (1975)
- Metropolis et al. (1973)

- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- period-doubling cascade @ low R
- windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)
- small copies of object embedded in it (fractal)

Fractals

- non-integer Hausdorff dimension
- self-similar

Examples: Cantor set, coastlines, trees, lungs, clouds, drainage basins, …

The Mandelbrot set

Fractals in computer graphics

www.youtube.com/watch?v=G_GBwuYuOOs

Matthew Ward, WPI
davis.wpi.edu/~matt/courses/fractals/trees.html
Fractals in maps

Newton’s method on \( x^2 - 1 = 0 \)

Fractals and chaos...

The connection: many (most) chaotic systems have fractal state-space structure.

But not “all.”

So far: mostly about maps.

• discrete time systems:
  • time proceeds in clicks
  • “maps”
  • modeling tool: difference equation

Next up: flows

• continuous time systems:
  • time proceeds smoothly
  • “flows”
  • modeling tool: differential equations

Attractors

• Attractors exist only in dissipative systems!
• Dissipation \(\leq\) contraction of state space under the influence of the dynamics
• Can still have chaos if no dissipation...just not chaotic attractors
Conditions for chaos in continuous-time systems

Necessary:
- Nonlinear
- At least three state-space dimensions (NB: only one needed in maps)

Necessary and sufficient:
- “Nonintegrable”
  i.e., cannot be solved in closed form

Concepts: review
- State variable
- State space
- Initial condition
- Trajectory
- Attractor
- Basin of attraction
- Transient
- Fixed point (un/stable)
- Bifurcation
- Parameter

A Lorenz applet:

www.exploratorium.edu/complexity/java/lorenz.html

(Note: by Jim Crutchfield, who will be here next week)
• Equations:
  \[ x' = a(y-x) \]
  \[ y' = rx - y - xz \]
  \[ z' = xy - bz \]

(first three terms of a Fourier expansion of the Navier-Stokes eqns)

• State variables:
  • \( x \) convective intensity
  • \( y \) temperature
  • \( z \) deviation from linearity in the vertical convection profile

• Parameters:
  • \( a \) Prandtl number - fluids property
  • \( r \) Rayleigh number - related to \( \Delta T \)
  • \( b \) aspect ratio of the fluid sheet

\[ x' = 16(y-x) \]
\[ y' = 45x - y - xz \]
\[ z' = xy - 4z \]
Maybe add Donny’s Lorenz movie here?

- See student work folder in directory above
- Courtesy of Donny Warbritton

Attractors

**Four types:**
- fixed points
- limit cycles *(aka periodic orbits)*
- quasiperiodic orbits
- chaotic attractors

A nonlinear system can have any number of attractors, of all types, sprinkled around its state space.

Their basins of attraction (plus the basin boundaries) partition the state space.

And there’s no way, *a priori*, to know where they are, how many there are, what types, etc.
Attractors

- Fixed point

- Limit cycle

- Quasi-periodic orbit...

“Strange” or chaotic attractors

- Often fractal
- Covered densely by trajectories
- Exponential divergence of neighboring trajectories...

Lyapunov exponents

- Nonlinear analogs of eigenvalues: one $\lambda$ for each dimension

- Negative $\lambda_i$ compress state space; positive $\lambda_i$ stretch it
- $\Sigma \lambda_i < 0$ for dissipative systems
- Long-term average in definition; biggest one dominates as $t \rightarrow$ infinity
- Positive $\lambda_i$ is a signature of chaos
- $\lambda_i$ are same for all ICs in one basin

Lyapunov exponents: summary
“Strange” or chaotic attractors:

- exponential divergence of neighboring trajectories
- often fractal
- covered densely by trajectories
- contain an infinite number of “unstable periodic orbits”…

Unstable periodic orbits (UPOs)

Attractor “bones”…

Poincare recurrence
The rest of today...

- Lunch (cafeteria downstairs)
- Dynamics Lab I: (here)
  - Meet here at 1:30pm
  - Bring your laptop, if you have one here
  - Lab handouts on the CSSS wiki
- 3pm — Intro to student projects (here)
- 4:15pm — Start thinking & talking about those projects!
Moral: numerical methods can run amok in “interesting” ways…
• can cause distortions, bifurcations, etc.
• and these look a lot like real, physical dynamics…
• source: algorithms, arithmetic system, timestep, etc.
• Q: what could you do to diagnose whether your results included spurious numerical dynamics?

So ODE solvers make mistakes.
…and chaotic systems are sensitively dependent on initial conditions….

Shadowing lemma
Every* noise-added trajectory on a chaotic attractor is shadowed by a true trajectory.

Important: this is for state noise, not parameter noise.
(*) Caveat: not if the noise bumps the trajectory out of the basin

Solving PDEs
Section

Not the same thing as a projection!

The driven damped pendulum

What bifurcations look like on a Poincare section

The Lorenz attractor

Cantor set!
(remember: not always...)
What about a section of a UPO?

Aside: finding UPOs

- Section
- Look for close returns
- Cluster
- Average
- See Gunaratne, So papers

Computing sections

- If you’re slicing in state space: use the “inside-outside” function
- If you’re slicing in time: use modulo on the timestamp

Time-slice sections of periodic orbits:
some thought experiments

- pendulum rotating @ 1 Hz and strobe @ 1 Hz?
- pendulum rotating @ 1 Hz and strobe @ 2 Hz?
- pendulum rotating @ 1 Hz and strobe @ 3 Hz?
- pendulum rotating @ 1 Hz and strobe @ 1/2 Hz?
- pendulum rotating @ 1 Hz and strobe @ π Hz? (or some other irrational)

Stability, $\lambda$, and the un/stable manifolds
These $\lambda$ & manifolds play a role in control of chaos...

Lyapunov exponents:
- one $\lambda$ for each dimension; $\sum \lambda < 0$ for dissipative systems
- $\lambda$ are same for all ICs in one basin
- negative $\lambda$ compress state space along stable manifolds
- positive $\lambda$ stretch it along unstable manifolds
- biggest one ($\lambda_1$) dominates as $t \to \infty$
- positive $\lambda$, is a signature of chaos

calculating them:
- From equations: eigenvalues of the variational matrix (see variational system notes on CSCI5446 course webpage; see link from Liz’s homepage.)
- From data: various algorithms that are hideously sensitive to numerics, noise, data length, & algorithmic parameters.

Calculating $\lambda$ (& other invariants) from data

- A good reference: Kantz & Schreiber, *Nonlinear Time Series Analysis* (Abarbanel’s book is also very good)
- Associated software: TISEAN
  [www.mpipks-dresden.mpg.de/~tisean](http://www.mpipks-dresden.mpg.de/~tisean)

**TISEAN**
Nonlinear Time Series Analysis
Rainer Hegger
Holger Kantz
Thomas Schreiber

Go to Version 3.0.1 (released March 2007)
Go to Version 2.1 (released December 2000)

**TISEAN 3.0.1: Table of Contents**

Generating time series

A few routines are provided to generate test data from simple equations. Since these are powerful packages like the Matlab Trace and Eta Tools that can generate chaotic data, we have only included a valuable data-set that...

**Description of the program: lyup_k**

The program estimates the largest Lyapunov exponent of a given scalar data set using the algorithm of Kantz.

**Usage:**

```
lyup_k [options]
```

Everything not being a valid option will be interpreted as a potential filename. Given no filename at all, mean-mad stats. Also: no-mean stats.
Kantz’s algorithm:

1. Choose point K
2. Look at the points around it
3. Measure how far they are from K
4. Average those distances
5. Watch how that average grows with time ($\Delta n$)
6. Take the log, normalize over time $\Rightarrow S(\Delta n)$
7. Repeat for lots of points K and average the $S(\Delta n)$

If you’re lucky:

The slope of the scaling region—iff one exists—is the $\lambda_1$

Calculating $\lambda$ (& other invariants) from data

- Be careful! TISEAN has lots of knobs and its results are incredibly sensitive to their values!
- Use your dynamics knowledge to understand & use those knobs intelligently
- Look at the plot; do not blindly fit a regression line to something that has no scaling region

Fractal dimension:

- Capacity
- Box counting
- Correlation ($d_2$ in TISEAN)
- Lots of others:
  - $K$th nearest neighbor
  - Similarity
  - Information
  - Lyapunov
  - ...
- See Chapter 6 and §11.3 of Kantz & Schreiber
But often you can’t.

How to undo a projection?

Delay-coordinate embedding

“reinflate” that squashed data to get a topologically identical copy of the original thing.

Reconstruction space

Mechanics

TISEAN’s delay command does this.
Measure just the $x$ coordinate…

…and then embed:

Takens' theorem

For the right $\tau$ and enough dimensions, the embedded dynamics are diffeomorphic to (have same topology as) the original state-space dynamics.

* Whitney, Mane, …

Note: the measured quantity must be a smooth, generic function of at least one state variable, and must be uniformly sampled in time.

Diffeomorphisms and topology

Diffeomorphic: mapping from the one to the other is differentiable and has a differentiable inverse.

What that means:
- qualitatively the same shape
- have same dynamical invariants (e.g., $\lambda$)

Picking $\tau$

TISEAN contains tools that help you do this (e.g., mutual)

Picking $m$

$m > 2d$: sufficient to ensure no crossings in reconstruction space:

…may be overkill.

“Embedology” paper: $m > 2d_{\text{box}}$
(box-counting dimension)

TISEAN contains tools that help you do this (e.g., false_nearest)
NLTSₐ* of computer performance dynamics

* nonlinear time-series analysis

bzip₂ dynamics on an Intel Core2

\( \lambda = 1.06 \)

\( \tau = 194, m = 10 \)

Mytkowicz et al., Chaos 19:033124

bzip₂ dynamics on an Intel Pentium 4

\( \lambda = 0.07 \)

\( \tau = 973, m = 12 \)

Mytkowicz et al., Chaos 19:033124

povray dynamics on an Intel Core2

\( \lambda = 0.04 \)

\( \tau = 930, m = 8 \)

Mytkowicz et al., Chaos 19:033124

Caveat: need enough data...

If \( \Delta t \) is not uniform

Theorem (Takens): for \( \tau > 0 \) and \( m > 2d \), reconstructed trajectory is diffeomorphic to the true trajectory.

Conditions: evenly sampled in time

Interspike interval embedding

idea: lots of systems generate spikes — hearts, nerves, etc.

if you assume that the spikes are the result of an integrate-and-fire system, then the \( \Delta t \) has a one-to-one correspondence to some state variable’s integrated value…

in which case the Takens theorem still holds.

(with the \( \Delta ts \) as state variables)

Sauer Chaos 5:127

What if we measured time-series data from a roulette wheel?

The Eudaemonic Pie
(or The Newtonian Casino)
The Santa Fe competition

- Weigend & Gershenfeld, 1992
- put a bunch of data sets up on an ftp server
- and invited all comers to predict their future
- chronicled in *Time Series Prediction: Forecasting the Future and Understanding the Past*, Santa Fe Institute, 1993 (from which the images on the following half-dozen slides were reproduced)

The Santa Fe competition: data

- Laboratory laser
- Medical data (sleep apnea)
- Currency rate exchange
- RK4 on some chaotic ODE
- Intensity of some star
- A Bach fugue

Embedding + patch models: (Sauer)

Neural net: (Wan)
Further out:

Sauer

Lorenz’s method of analogues

Wan

A $k$-nearest neighbor modification of LMA
Using $k$LMA to predict computer dynamics

![Graph showing column major cache misses with RMSPE = 11.566 (0.07% of the average)](image)

**Garland/Brady Intelligent Data Analysis 2011**

**Noise...**

Linear filtering: a bad idea if the system is chaotic

Nonlinear alternatives:
- use the stable and unstable manifold structure on a chaotic attractor...

**Farmer & Sidorowich, in Evolution, Learning and Cognition, World Scientific, 1983**

**Idea:**
- If you have a model of the system, you can simulate what happens to each point in forward and backward time
- If your system has transverse stable and unstable manifolds, that does useful things to the noise balls
- Since all three versions of that data should be identical at the middle time, can average them
  - noise reduction!
  - Works best if manifolds are perpendicular, but requires only transversality

**Results:**

![Graph showing signal and predicted signal with error between signals](image)

**Farmer & Sidorowich, in Evolution, Learning and Cognition, World Scientific, 1983**

**Noise...**

Linear filtering: a bad idea if the system is chaotic

Nonlinear alternatives:
- use the stable and unstable manifold structure on a chaotic attractor
- use the topology of the attractor...
Computational Topology

Why: this is the fundamental mathematics of shape, complements geometry.

What: compute topological properties from finite data

How:
• introduce resolution parameter
• count components and holes at different resolutions
• deduce topology from patterns therein


Connectedness: definitions

• how many “lumps” in a data set:
• \( \varepsilon \)-connectedness (after Cantor)
• \( \varepsilon \)-connected components
• \( \varepsilon \)-isolated points:

Connectedness: examples

If the data points are samples of a disconnected fractal like this:
The number of connected components looks like this:

Robins et al., Physica D 139:276, Nonlinearity 11:913

(note obvious tie-in to fractal dimension…)

Connectedness and filtering

The effect of noise is to add isolated points to the set and a shoulder to the \( C(\varepsilon) \) curve:

So if you know that the object is connected — like the attractor of a flow — you can reasonably assume that any isolated points are noisy, and remove them by pruning with \( \varepsilon = \varepsilon^* \)

Robins et al., Intelligent Data Analysis 8:505, Chaos 14:305

Connectedness: examples

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Connectedness: examples

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Robins et al., Physica D 139:276, Nonlinearity 11:913

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Continuity and filtering

Motivation:
• deterministic, differentiable dynamics (maps & flows) are continuous

Conjecture:
• if the image of a connected set is not connected, more than one dynamics is at work

Approach:
• track connectedness over time

Applications:
• pulling apart interleaved dynamics, removing noise…

Alexander et al., CHAOS, 2012
Chaos and control...

key concepts:
• dense attractor coverage
• exponential trajectory separation
• un/stable manifold structure
• local-linear control

Control:
getting from A to B, minimizing some cost functional...

Lorenz System:
denseness, reachability, and control

Denseness & reachability in a real engineering application

• can control position/volume/density of attractor — within limits
• possibly not reachable any other way
• not for time-critical applications (that “eventually”)

OGY control

• dense attractor coverage — reachability (*)
• un/stable manifold structure — controllability

* again, eventually…

Using Chaos to Broaden the Capture Range of a Phase-Locked Loop
Dietrich-Niklas, Ross, 2011

Ogilvie et al., PRE 88:1196
• dense attractor coverage $\rightarrow$ reachability
• un/stable manifold structure $\rightarrow$ controllability
• exploit sensitive dependence, too???
  $\implies$ “targeting”

**Lorenz System:**
SDOIC-based targeting

OGY & co. have been used in tons of systems; see Shinbrot review paper.
Alfred Hubler has done a lot of cool stuff in this area as well.

Four R switches; 240X faster

**Other cool ways to use invariant manifolds**

Want to get a spacecraft onto a “halo orbit,” which is a UPO of the dynamics.

Unstable Periodic Orbits (UPOs) have invariant manifolds:

• Stable Invariant Manifold ($W^S$)
  – The set of all trajectories a particle could use to arrive onto the UPO.

• Unstable Invariant Manifold ($W^U$)
  – The set of all trajectories a particle could take after a small perturbation from the UPO.

**Low-energy (cheap) orbit transfers**

• Depart along $W^s_{L1}$ & arrive on $W^s_{L2}$

**Homoclinic orbits - The best case**

• If a trajectory in Stable and Unstable intersect (“homoclinic connection”)
Can we do any of that in spatially extended systems? (i.e. harness the butterfly effect, exploit un/stable manifold geometry?)

Sensitive flames (1856 – 1930s)

I repeat a passage from Shakespeare:

"The story heard but half of beauty knows,
Like a broad table void itself disposed.
For love his lofty triangle is angulous.
And with the writhings of the great godhead,
An inward organ the abominations
Put there their swelling mass, and when the spark
Starts, words, like struggling honey in a cell,
And through the pores and veins softly leaks.
A silver veil, which heaven alone meant to make."

The flame selects from the sounds those to which it can respond. It notices none by the slightest nod, to others it knows more intimately, to some its influence is very profound, while to many sounds it turns an entirely deaf ear.

A 2D jet

Forcing the jet flow

MEMS actuators

End view

Video: overhead view at 2Hz, 10V

Area of individual flap is 1.0 x 0.25mm

Peacock et al., Exp. Fluids 37:22

The Butterfly effect in action…

- No forcing
- 6Hz forcing

Forcing generates coherent structures that enhance entrainment and mixing

Does this have anything to do with reality?

Measurement & isolation:

Communication and chaos:

- Two coupled Lorenz systems will synchronize
- Robust to a small amount of noise
- Use this to transmit & receive information

\[
\begin{align*}
x' &= a(y-x) \\
y' &= r(x + \varepsilon x) - y -xz \\
z' &= xy - bz
\end{align*}
\]

- Chaotic carrier wave, so hard to intercept or jam

Another interesting application:
chaos in the solar system

- Orbits of Pluto, Mars
- Kirkwood gaps
- Rotation of Hyperion & other satellites
- ...

Solar system stability:

- Recall: two-body problem not chaotic
- But three (or more) can be…

Pecora & Carroll, Phys. Rev. Lett 64:461
Exploring that issue, circa 1880:

An orrery, which is a mechanical computer whose gear ratios and circular platters simulate the orbits of the planets.

Exploring that issue, circa 1980:

- write the $n$-body equations for the solar system
- solve them using symplectic ODE solvers on a special-purpose computer

The digital orrery (Wisdom & Sussman)

Should we worry?

- No.

Kirkwood gaps:

Chaos and the Kirkwood gaps

Evidence in favor of the conjecture:


Chaotic tumbling of satellites:

Voyager and Galileo saw this…

Ap. J. 97:570

Chaotic tumbling of satellites:

This happens for all satellites at some point in their history, unless they are perfectly spherical and in perfectly circular orbits (pf: KAM theorem; see Wisdom paper on syllabus)

Some of them are still tumbling chaotically because of their geometry, but most (like the earth and its moon) have settled down into tidal equilibria

More chaos in the solar system:

• obliquity of Mars (Touma & Wisdom, Science 259:1294)

• etc.

Musical Variations from a Chaotic Mapping

Pitch sequence:

Dabby Chaos 6/95


www.solarviews.com

Also see https://www.youtube.com/watch?v=82lsxGPy3GA.
Chaotic variations on movement sequences

original piece

chaotic mapping

chaotic variation

Bradley & Stuart, Chaos 8:800
Lorenz

Rossler

original cell
itinerary

variation
cell
itinerary

medley—original

Rossler variation of medley

random variation of medley
original cell itinerary  

variation cell itinerary

Interpolation

Corpus-based approach

• graph captures motions of one joint
• note: specific to the genre of the corpus!
Initial state

Target state

Graph search

...for 44 joints in parallel!

initial

target
Con/cantation: (chaotic variations)
A computer-assisted theme and variations performance project

synchronousobjects.osu.edu/content.html