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Small-world effect induced by weight randomization on regular networks

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Abstract

The concept of edge weight provides additional depth for describing and adjusting the properties of networks. Redistribution of edge weight can effectively change the properties of networks even though the corresponding binary topology remains unchanged. Based on regular networks with initially homogeneous dissimilarity weights, random redistribution of edge weight can be enough to induce small world phenomena. The effects of random weight redistribution on both static properties and dynamical models of networks are investigated. The results reveal that randomization of weight can enhance the ability of synchronization of chaotic systems dramatically.
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Introduction

Network analysis is now popularly used to describe relationships and collective behavior in many fields [1–3]. A binary network has a set of vertices and a set of edges that represent the relationship between any two vertices. As the edge represents only the presence or absence of interaction, such an approach is limited in its ability to describe relations with different strengths or having a variety of levels. Interaction intensity (described by edge weight) usually plays an important role in many real networks and it therefore becomes necessary to take edge weight into account and to study weighted networks. For example, the number of passengers or flights between any two airports in airport networks [4–6], the closeness of any two scientists in scientific collaboration networks [6–9], and the reaction rates in metabolic networks [10] are all crucial to characterize the corresponding networks. Recently more and more studies in complex networks have focused on weighted networks. The problems studied involve the definition of edge weight and other quantities which characterize the weighted networks [11–13], the empirical studies about its statistical properties [4–9,14,15], evolving models [16–22], and transportation or other dynamics on weighted networks [23–25].

From Watts and Strogatz's model of small-world networks [26], we know that rewiring of edges can significantly change properties of networks. For weighted networks, the redistribution of edge weight provides another way to adjust network structure. What significant changes of network properties, however, will be induced by redistribution of edge weight? Up to date, this question has not been thoroughly investigated. We can imagine that rewiring of edges will be the dominant factor. Therefore, at first sight, theoretically it looks like study of randomization of edge weight is not quite relevant. However, in some real networks, the cost of rewiring edges is much larger than the cost of redistributing edge weights. Here cost could consist of money, human resource, time and so on. For example, let us consider network of products, where vertices represent products and edges denote technologies that produce products from others. One of the central topic in Economics is to optimize the distribution of resource to achieve the maximum efficiency. In the angle of product network, this is to either adjust the connection between products by developing new technolo-

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gies, or change edge weight by assigning different amount of resources to existing technologies. In a long term, it is obvious that the investment on research and development, which rewires the edges, boosts the development of society. However, it is costly. In a short time, we can mainly focus on redistribution of resources, which changes the edge weight, to improve the efficiency. So understanding the effects of randomizing edge weight on properties of networks could provide some insight to above question.

In this Letter, we discuss the effects of randomizing weight on properties of weighted networks. In our previous work [9], we introduced one method to disturb the weight-edge correspondence and investigated its effects on network structures of empirical networks. There we focused on static network structures such as shortest path and betweenness. Here, instead of an uncontrollable empirical network, we start from a regular network, as in the small-world model [26]. In order to discuss the effect of randomizing edge weight alone, we have to separate those two factors, namely rewiring of edges and randomization of edge weights. Therefore we will do two versions of randomizing edge weights: First without rewiring (regular networks), and second after rewiring of edges (small-world networks). Then we will compare static properties of the networks before and after the redistribution of edge weights. Then for the first version, we will investigate its effects on chaotic synchronization of Logistic maps.

Model: Redistribution of edge weight

In weighted networks, edge weight can be presented either as measures of dissimilarity or similarity. For dissimilarity weight w, e.g. the distance between two airports, the distance between two vertices connected by a third node and two edges (with dissimilarity weights w_1 and w_2 respectively) is $l = w_1 + w_2$. For similarity weight $\tilde{w}$, e.g. the number of cooperation between any two scientists in scientific collaboration networks, the calculation of similarity distance between two vertices connected by a third node and two edges (with similarity weights $\tilde{w}_1$ and $\tilde{w}_2$ respectively) is $\tilde{w} = 1/(\frac{1}{\tilde{w}_1} + \frac{1}{\tilde{w}_2})$, which is smaller than both $\tilde{w}_1$ and $\tilde{w}_2$. Distance, then, can be defined as the inverse of similarity weight, namely $l = 1/\tilde{w}$. In the following discussion, the dissimilarity weight is chosen as $w_{ij} \in [1, \infty)$, and consequently the similarity weight $\tilde{w}_{ij} = 1/w_{ij}$ is in $(0, 1]$. In this Letter, we set dissimilarity weight on edges, and besides the calculation of clustering coefficient, we stick with dissimilarity weight.

Our initial setup is a ring lattice with N vertices, k edges per vertex and each edge having the same dissimilarity weight $w = 5$. All of connections are undirected. Here, we assume that there is a minimum unit of weight $\Delta w = 1$. Starting from this original network, we define the procedure of redistributing edge weight as follows for version one:

(1) Every unit of weight in the original lattice is removed, with probability p, from the original edge and transferred to a edge randomly chosen from the whole lattice.
(2) Step 1 is repeated until each unit of weight in the original lattice has been tried once. The reallocated weights will not be considered again.
(3) If the unit of weight being considered is the only unit of weight left on that edge, it will not be moved. This is to avoid disconnecting a edge so that we have the same corresponding binary network.

For version two:

(1) For a certain probability P, rewiring edges as in small-world models [26].
(2) Repeat version one procedure.

For version one, without changing the binary structure, with $p \in [0, 1]$, this construction allows us to adjust a regular network from one with uniform edge weight ($p = 0$) to one with random weight distribution ($p = 1$). For version two, it allow us to see the effect of randomizing edge weight after rewiring of edges. Before we discuss and compare quantities such as distance and clustering coefficients for networks before and after edge weight redistribution, we first need to make them comparable. In a binary network all edge weights are set as 1, while in weighted networks there is no such natural prescription. Different networks can have different measures and even different researchers could use different measures. Therefore we first need to normalize edge weight to provide a common reference for all edge weights.

Normalization of edge weight

For similarity weight, we would like to choose $\tilde{w} = 1$ as the reference and put all similarity edge weights into [0, 1], indicating $w \in [1, \infty)$ for dissimilarity weight. One possible way, which has been used by several authors [27–29], is to normalize edge weights by their maximum $\tilde{\omega}_i = \frac{\tilde{w}_i}{\max\{\tilde{w}_1, \tilde{w}_2, \ldots\}}$. In this way, $\tilde{\omega} \in [0, 1]$ and networks coming from different backgrounds are now comparable. However, this normalization has a disadvantage. For example, imagine two scientific collaboration networks, A and B, with the same topological connection but correspondingly every edge weight $\tilde{w}_i^a = 2\tilde{w}_i^b$. After the above normalization, we will get two identical networks so that every quantity from them will be the same, which in a sense is right. However, one could easily say that overall scientists in network A have closer relation than those in B so that the clustering coefficient $C^a = 2C^b \neq C^b$. Here we see that some information has been lost in this maximum-based normalization (MBN).

Our suggestion for normalization is to: (1) try to define the measure of a relationship from its background and put it into [0, 1] in the first place, without applying MBN. (2) If the original measures are not possible to do the above, take their predefined weight and apply MBN.

For example, for scientific collaboration network we use the *tanh* function to convert number of collaboration or citation into a measure of relationship that will be in [0, 1] [9]. Sometimes in such a realization there could be no edge with weight 1, but following our suggested normalization procedure, we would not do another MBN. For cases that lack such good measure, our approach comes back to MBN. In the following calculation, we always keep dissimilarity weight in $[1, \infty)$ and similarity weight in $[0, 1]$.

Effects on static properties of network

After redistributing the dissimilarity weights over the whole network, the weighted distance of a path can be easily calculated from the sum of dissimilarity weights for any given path. For the calculation of weighted clustering coefficient, however, we must define a quantity to measure the closeness of connections within i's neighborhood. It is not a trivial question, so we will discuss its calculation firstly.

Several definitions for the weighted clustering coefficient have already been suggested [27–29]. Holme has argued that this quantity should fulfill four requirements and provided a definition of weighted clustering coefficients based on the similarity weight [29]

$$\mathbf{C}_H^w(i) = \frac{\sum_{j,k} \tilde{w}_{ij}\tilde{w}_{jk}\tilde{w}_{ki}}{(\max_{lm} \tilde{w}_{lm}) \sum_{j,k} \tilde{w}_{ij}\tilde{w}_{ki}}, \tag{1}$$

where in the denominator $\max_{lm} \tilde{w}_{lm}$ is the possible maximum edge weight and $\tilde{w}_{jk}$ are all the various edge weights. Onnela generalized the Watts–Strogatz clustering coefficient as [28]

$$\mathbf{C}_O^w(i) = \frac{2}{k_i(k_i - 1)} \sum_{j,k} \frac{(\tilde{w}_{ij}\tilde{w}_{jk}\tilde{w}_{ki})^{1/3}}{\max_{lm} \tilde{w}_{lm}}. \tag{2}$$

This definition also fulfills the requirements proposed by Holme [29]. In both definitions $\max_{lm} \tilde{w}_{lm}$ is used to normalize edge weights as mentioned in MBN. In our calculation, because we have already normalized edge weight, we use the following equation,

$$C_H^w(i) = \frac{\sum_{j,k} \tilde{w}_{ij}\tilde{w}_{jk}\tilde{w}_{ki}}{\sum_{j,k} \tilde{w}_{ij}\tilde{w}_{ki}} \tag{3}$$

and

$$C_O^w(i) = \frac{2}{k_i(k_i - 1)} \sum_{j,k} (\tilde{w}_{ij}\tilde{w}_{jk}\tilde{w}_{ki})^{1/3}. \tag{4}$$

Similar to the WS small-world network, the average weighted path length $L(p)$ and the weighted clustering coefficient $C(p)$ are used to present the structural properties of the network. Fig. 1 reveals that with the above random redistribution of edge weight, the average path length decreases, while the average clustering coefficient increases. This demonstrates that besides rewiring edge, randomizing edge weight also leads to small-world phenomenon.

We have also studied this effect after rewiring of edges. We firstly rewire edges with certain probability P, and then redistribute the edge weight. Now we can investigate the effect of randomizing edge weights on properties of the networks after rewiring of edges. As shown in Fig. 2, the effect of redistribution is not as significant as in the former case, but it is still obvious. It indicates that randomizing edge weights could still affect the properties of small-world networks. This implies that redistribution of weight is complementary to rewiring. Considering the discussion about their costs as in product network in the introduction part, those two plots do suggest the value of weight redistribution.

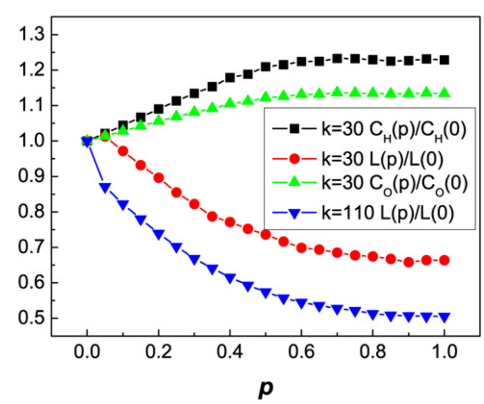

Fig. 1. Without rewiring edges, characteristic path length $L(p)/L(0)$ and clustering coefficient $C(p)/C(0)$ for the family of randomly weight redistributed networks ($N = 200$). k is the edges per vertex and C_H and C_O are defined in Eq. (3) and Eq. (4) respectively. The x-axis is the probability of redistributing edge weight, and the y-axis is the value of $L(p)/L(0)$ and $C(p)/C(0)$, where $L(0)$ and $C(0)$ are values before randomizing edge weight. All results are averaged over 20 random realizations of the redistribution process and the relative standard deviation is less than 2%.

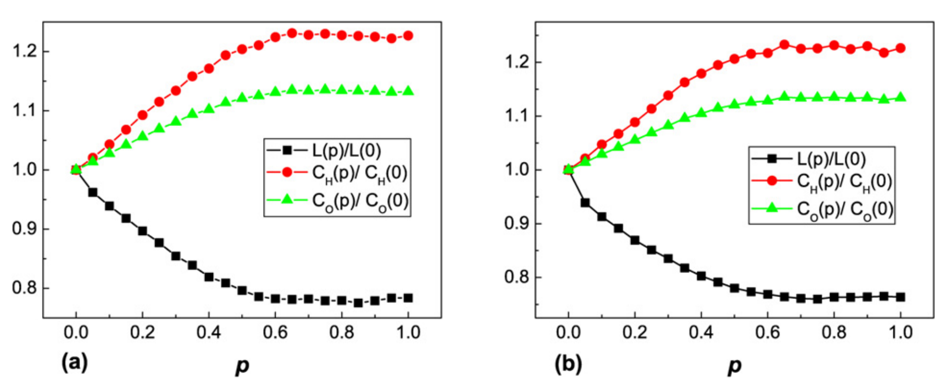

Fig. 2. After rewiring edges with certain probability $P = 0.05$ (a) and $P = 0.2$ (b), characteristic path length $L(p)/L(0)$ and clustering coefficient $C(p)/C(0)$ for the family of randomly weight redistributed networks ($N = 200, k = 30$). All results are averaged over 20 random realizations of the redistribution process and the relative standard deviation is less than 2%.

We also investigate the influence of network size and density of edges on the effects of weight redistribution. It seems that the effects of weight redistribution described above are relatively more significant in dense networks than in sparse networks. For a given randomization probability p, Fig. 3 shows the results of $L(p)/L(0)$ as a function of $Nk/N(N-1)$. With increasing density of the network, $L(p)/L(0)$ decreases until it reaches a minimum. A similar conclusion holds for different network sizes.

Besides the average weighted path length and the weighted clustering coefficient, we also plot the distribution of vertex and edge betweenness after weight redistribution. The betweenness of vertices [30] and edges [31] in a network is defined to be the number of the shortest paths passing through them. We calculate betweenness as follows: First, we find all the shortest paths between any pair of nodes, including the cases with more than one shortest path. Second, we count the edge and vertex betweenness from the set of shortest paths. The redistribution of dissimilarity weight affects the shortest path, thereby changing the values of betweenness. As shown in Fig. 4, edge betweenness becomes a power-law and vertex betweenness turns into a Γ distribution. In the original network, both of them should take only a few value since the original network is homogenous. The distributions of vertex/edge betweenness in small-world model [26] also changes into power-law/Γ distribution correspondingly with $N = 1000, k = 10$ and $P = 0.02$. These changes again imply that we have changed the global structural features even though edge weight was redistributed merely.

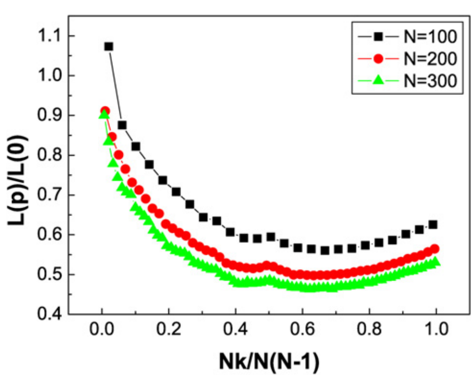

Fig. 3. With a given random redistribution probability $p = 1$, $L(p)/L(0)$ is plotted as a function of network density, described by $Nk/N(N-1)$. Similar results appear for different N.

Effects on synchronization of logistic maps

In above part, we showed the small-world effects on static properties of networks due to the random redistribution of weight. Next we investigate its effects on synchronization of chaotic system on networks.

Since their introduction in 1989 [32], coupled maps have been used as paradigmatic examples in the study of the emergent behavior of complex systems as diverse as ecological networks, the immune system, or neural and cellular networks. In recent years, the synchronization of chaotic system on complex networks has drawn much attention. In [33,34] and other works, chaotic system on randomly coupled maps and other network architectures is investigated. It is found that the synchronization properties of a system are strongly dependent on its particular architecture. Networks with the same number connectivity might have very different collective behavior. Earlier work mainly focused on the effect of network topology on synchronization. Here, we investigate regular networks of chaotic maps connected symmetrically and mainly focus on the influence of redistributing weight on synchronization. We take the following coupled maps

$$x_i(t+1) = (1-\varepsilon_i) f\big(x_i(t)\big) + \frac{\gamma}{m} \sum_{i \neq j} J_{ij} f\big(x_j(t)\big), \tag{5}$$

where $x_i(t)$ is a state variable and t denotes the discrete time. $f(x)$ prescribes the local dynamics, and is chosen as the logistic map $f(x_i) = \alpha x_i(1-x_i)$ with $\alpha = 3.9$. $\varepsilon_i = \frac{\gamma}{m} \sum_{i \neq j} J_{ij}$ giving the long-range coupling strength. The sum is taken over all the m coupling nodes with i. Here we assume coupling of the nearest and next nearest neighbors, so m is the number of nearest-neighbors and next nearest-neighbors of any node. Given the dissimilarity weight w_{ij} between any two nodes i and j connected directly, we define the following edge-preferential interaction: The interaction J_{ij} reads $1/w_{ij}$ for the nearest neighbors (even if they may be connected through a third node s), reads $1/\min_s(w_{is} + w_{sj})$ if i and j are next nearest-neighbors and

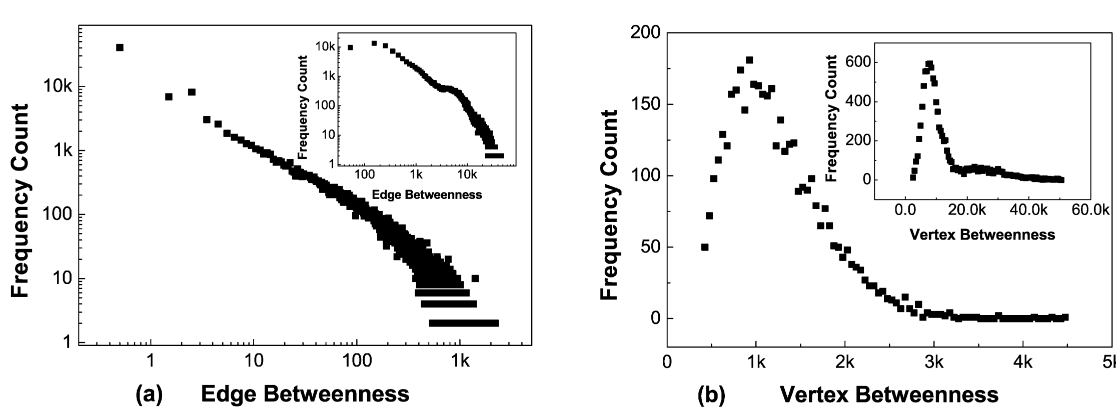

Fig. 4. Distribution of edge betweenness (a) and vertex betweenness (b) after the full random redistribution of weight ($p = 1$). The inset is the distribution of edge betweenness and vertex betweenness on WS small-world networks with the same parameters as in Ref. [26] and $P = 0.02$.

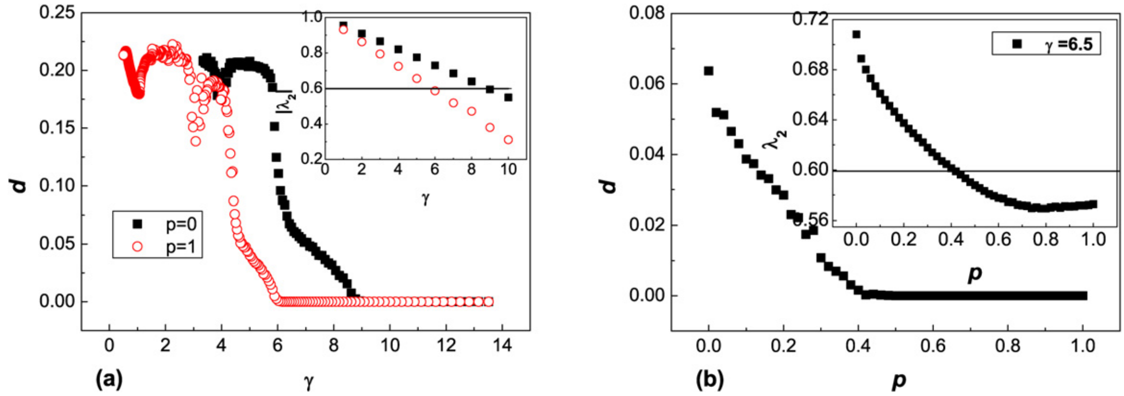

Fig. 5. (a) Final d as the function of γ. The inset is the second largest eigenvalue as a function of γ, where the horizontal line indicates that the Lyapunov exponent is 0. When λ_2 is below the horizontal line, it will reach a synchronized state. (b) The final d and the second largest eigenvalue (inset) as a function of probability, p, with $\gamma = 6.5$.

reads 0 for all other cases. We make use of the degree of synchronization, $d = \frac{1}{N}\sum_i^N |x_i - E(x)|$, to discriminate whether complete synchronization has been attained or not, where $E(x)$ is the expectation of $\{x_i\}$.

Starting from a ring lattice with $N = 300$ vertices, $k = 120$ edges per vertex, and each edge having the same dissimilarity weight $w = 10$ in the initial network, we redistribute each $\Delta w = 1$ randomly with probability p. In the simulation, from random initial conditions, we neglect 500 steps as the transitional process and take d as the average of following 1500 steps. We average all results over 50 runs from random initial conditions. We first compare the final degree of synchronization, d, of homogeneous ($p = 0$) and fully randomized ($p = 1$) weight distributions for a given average weight. The results shown in Fig. 5(a) demonstrate that redistribution of weight is helpful to the synchronization of the system. In addition, the second largest eigenvalue, λ_2, of the coupling matrix shows the stability of the synchronized state [35]. The second largest eigenvalue at $p = 1$ is smaller than that at $p = 0$ (shown as the inset of Fig. 5(a)), which indicates it is easier to arrive at a synchronized state. Fig. 5(b) shows the final degree of synchronization, d, and the second largest eigenvalue as a function of probability p. These results indicate that the randomization of weight enhances the ability of synchronization. For a sparse network (small k), although the system cannot reach complete synchronization, as shown in Fig. 6, we still find the second largest eigenvalue is decreasing so that the randomization of weight is helpful in stabilizing the synchronization state.

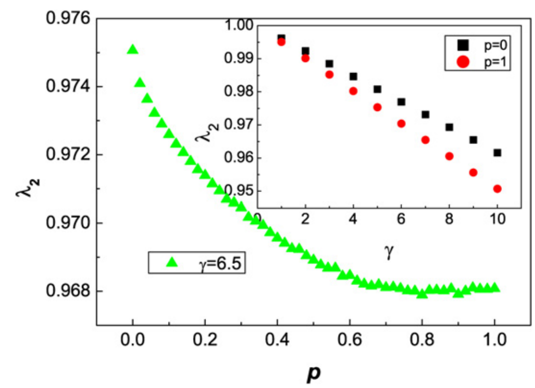

Fig. 6. The second largest eigenvalue as a function of p and γ (inset), with $k = 30$.

Conclusion

For weighted networks, redistribution of weight is an important way to change the properties of a network, besides changing its topology by rewiring of edges. In this Letter, we investigated the effects of redistribution of edge weight on static properties and on a dynamical model. We found random redistribution of edge weights could lead to small-world phenomena upon regular networks. Randomization of weight can reduce the average shortest path length while maintaining a high clustering coefficient. We also found that such redistribution enhances the ability of synchronization of coupled chaotic systems. Of course, we are not claiming that the small-world effect from redistribution of weights is as significant as that of rewiring of edges. It is, however, an important supplement to the small-world effect of rewiring. This shows that edge weight plays an important and relatively independent role on network properties. To know the correlations among dynamics, topology, and link weight distribution is crucial to understand the complex networks in real world.

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