Introduction to Nonlinear Dynamics

Santa Fe Institute

Complex Systems Summer School

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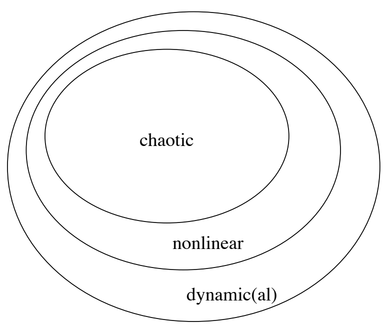
Chaos:

Complex behavior, arising in a deterministic nonlinear dynamic system, which exhibits two special properties:

- sensitive dependence on initial conditions
- characteristic structure…



Chaos:

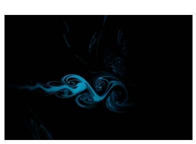
Complex behavior, arising in a deterministic nonlinear dynamic system, which exhibits two special properties:

- sensitive dependence on initial conditions
- characteristic structure…

Systems that exhibit chaos are ubiquitous; many of them are also simple, well-known, and "well-understood"

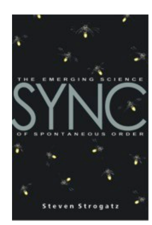
Where chaos turns up:

- Flows (of fluids, heat, …)
  - Eddy in creek
  - Weather
  - Vortices around marine invertebrates
  - Air/fuel flow in combustion chambers



Where chaos turns up:

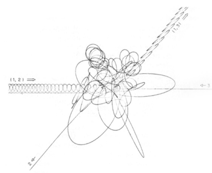
- Driven nonlinear oscillators
  - Pendula
  - Hearts
  - Fireflies



- and lots of other electronic, chemical, & biological systems

Where chaos turns up:

- Classical mechanics
  - three-body problem
  - paired black holes
  - pulsar emission
  - ….



Hut & Bahcall *Ap.J.* ***268****:319*

- Protein folding
- Population biology
- And many, many other fields (including yours)

- continuous time systems:
  - time proceeds smoothly
  - “flows”
  - modeling tool: differential equations
- discrete time systems:
  - time proceeds in clicks
  - “maps”
  - modeling tool: difference equation

A useful graphical solution technique:

- “cobweb” diagram
- *aka* return map
- *aka* correlation plot

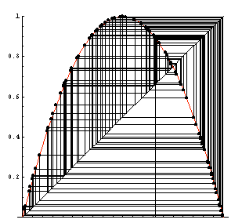


Image from Doug Ravenel’s website at URochester

Bifurcations

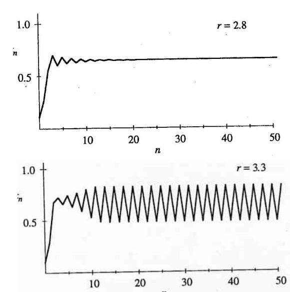
Qualitative changes in the dynamics caused by changes in *parameters*

Bifurcations

Qualitative changes in the dynamics caused by changes in *parameters:*

- Heart: pathology
- Eddy in creek: water level
- Olfactory bulb: smell
- Brain: blood chemicals
- etc. etc.

Bifurcations in the logistic map:

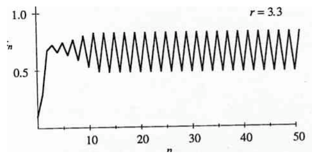


R=2.8

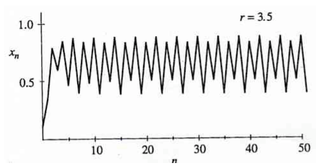
R=3.3

Note: in discrete time plots, it makes no sense to connect dots!!

Plots from *Strogatz*

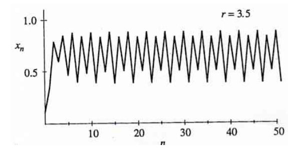


R=3.3

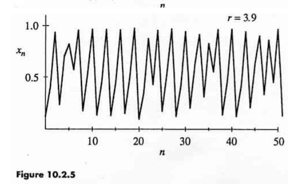


R=3.5

Plots from *Strogatz*

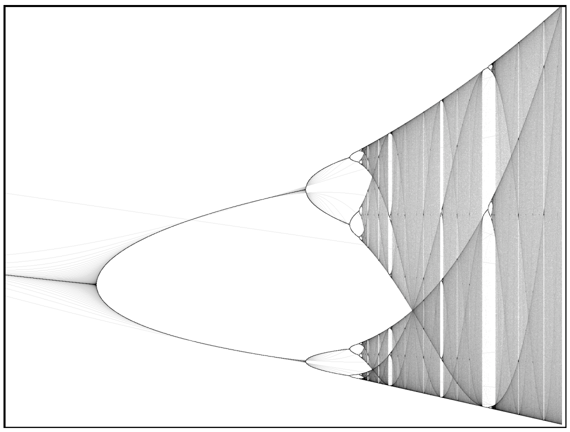


R=3.5

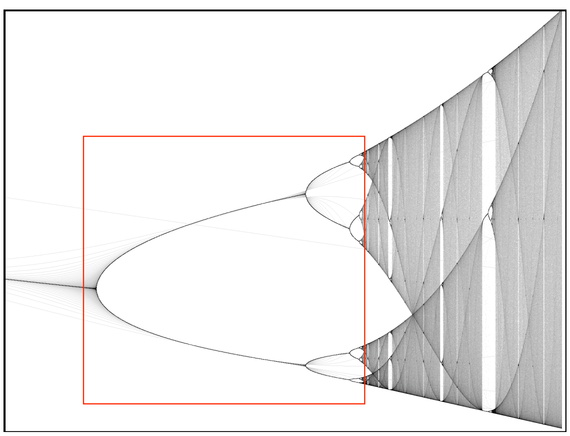


R=3.9

Plots from *Strogatz*

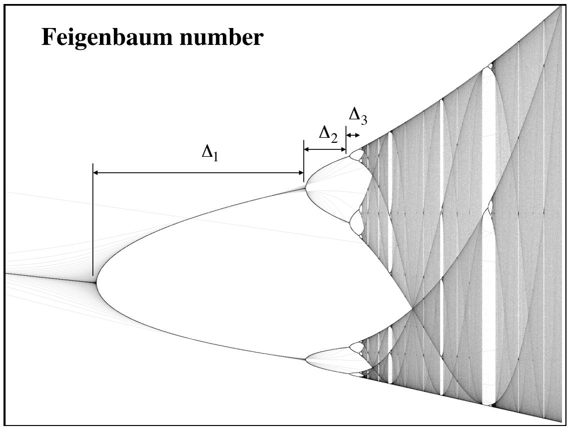


- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)



- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- *period-doubling cascade @ low R*

Feigenbaum number

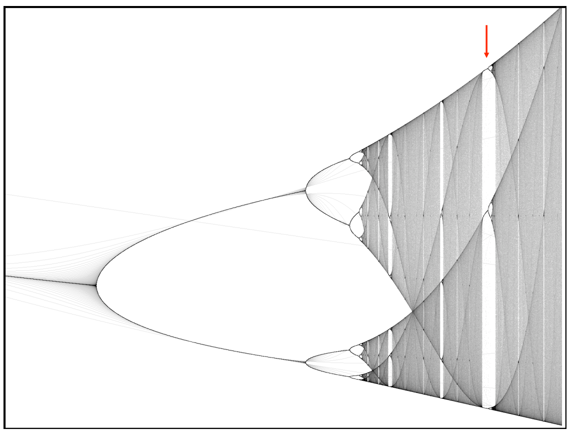


Universality!

Feigenbaum number and many other interesting chaotic/dynamical properties hold for any 1D map with a quadratic maximum.

Proof: renormalizations. See Strogatz §10.7

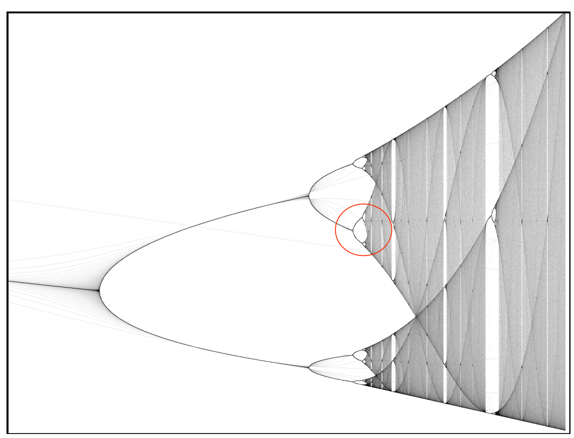
Don't take this too far, though…



- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- period-doubling cascade @ low R
- *windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)*

A bit more lore on periods and chaos:

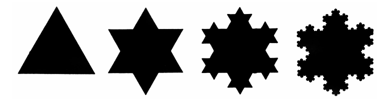
- Sarkovskii (1964)
- Yorke (1975)
- Metropolis *et al.* (1973)

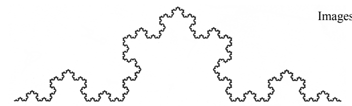


- chaos
- veils/bands: places where chaotic attractor is dense (UPOs)
- period-doubling cascade @ low R
- windows of order within the chaos, complete with their own period-doubling cascades (e.g., 3 to 6 to 12)
- *small copies of object embedded in it (fractal)*

Fractals

- non-integer Hausdorff dimension
- self-similar

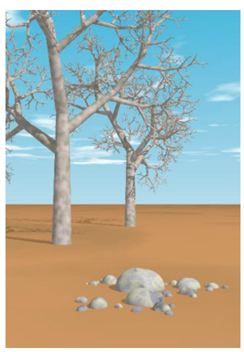




Images from Gleick

Examples: Cantor set, coastlines, trees, lungs, clouds, drainage basins, …

In computer graphics…



Matthew Ward, WPI
`davis.wpi.edu/~matt/courses/fractals/trees.html`

In maps:

Newton's method on $x^4 - 1 = 0$

From Strogatz

Fractals and Chaos…

The connection: *many (most)* chaotic systems have fractal state-space structure.

But **not** "*all*."

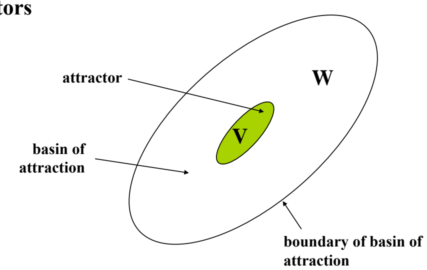
Yesterday was mostly about *maps.*

- discrete time systems:
  - time proceeds in clicks
  - "maps"
  - modeling tool: difference equation

Next: *flows.*

- continuous time systems:
  - time proceeds smoothly
  - “flows”
  - modeling tool: differential equations

Attractors



- Attractors exist only in dissipative systems!
- Dissipation ⟺ contraction of state space under the influence of the dynamics
- Can still have chaos if no dissipation…just not chaotic *attractors*

Conditions for chaos in continuous time systems:

__Necessary:__

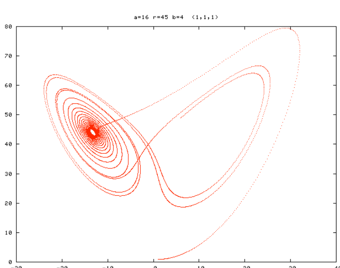
- Nonlinear
- At least three state-space dimensions

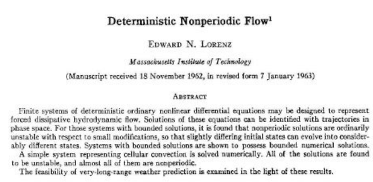
__Necessary and sufficient:__

- “Nonintegrable”
- i.e., cannot be solved in closed form

Concepts: review

- State variable
- State space
- Initial condition
- Trajectory
- Attractor
- Basin of attraction
- Transient
- Fixed point (un/stable)
- Bifurcation
- Parameter





Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

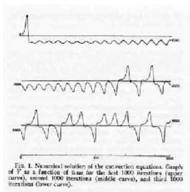
ABSTRACT

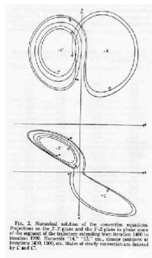
Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

J. Atm. Sci. **20:**130



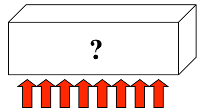


- Equations:

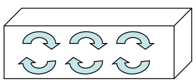
$x' = a(y-x)$

$y' = rx - y - xz$

$z' = xy - bz$

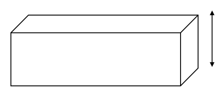


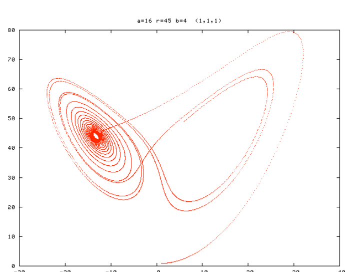
(first three terms of a Fourier expansion of the Navier-Stokes eqns)



- State variables:
  - *x* convective intensity
  - *y* temperature
  - *z* deviation from linearity in the vertical convection profile

- Parameters:
  - *a* Prandtl number - fluids property
  - *r* Rayleigh number - related to ΔT 
  - *b* aspect ratio of the fluid sheet





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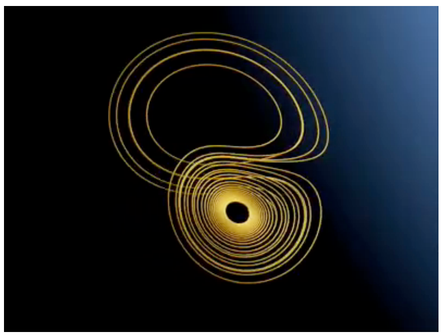
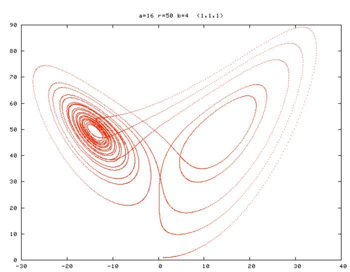
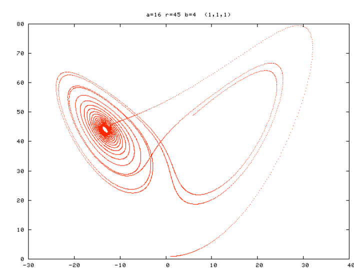
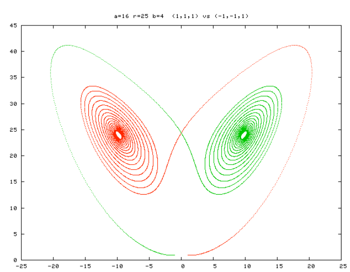
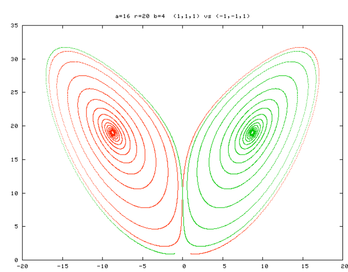
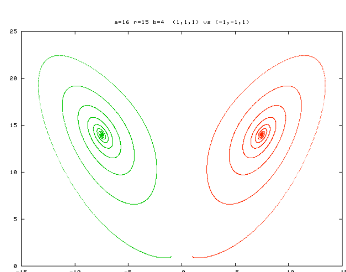












Attractors

Four types:

- fixed points
- limit cycles (*aka* periodic orbits)
- quasiperiodic orbits
- chaotic attractors

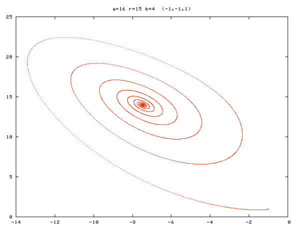
A nonlinear system can have any number of attractors, of all types, sprinkled around its state space

Their basins of attraction *partition* the state space

And there's no way, *a priori*, to know where they are, how many there are, what types, etc.

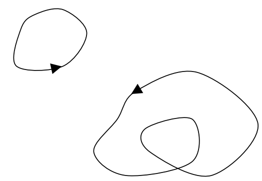
Attractors:

- Fixed point



Attractors:

- Limit cycle

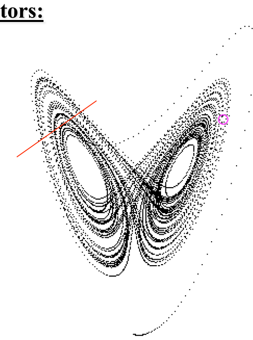


e.g., wall voltage!

Attractors:

- Quasi-periodic orbit…

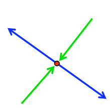
<u>“Strange” or chaotic attractors:</u>



- *often* fractal
- covered densely by trajectories
- exponential divergence of neighboring trajectories…

Lyapunov exponents:

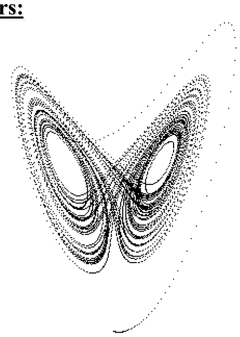
- nonlinear analogs of eigenvalues: one λ for each dimension



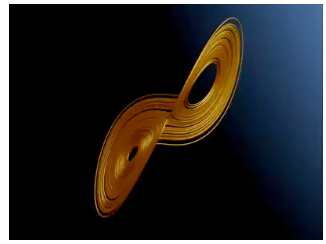
Lyapunov exponents:

- nonlinear analogs of eigenvalues: one λ for each dimension
- negative λ_i compress state space; positive λ_i stretch it
- $\Sigma\lambda_i < 0$ for dissipative systems
- long-term average in definition; biggest one dominates as $t \rightarrow$ infinity
- *positive λ is a signature of chaos*
- λ_i are same for all ICs in one basin

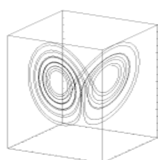
"Strange" or chaotic attractors:

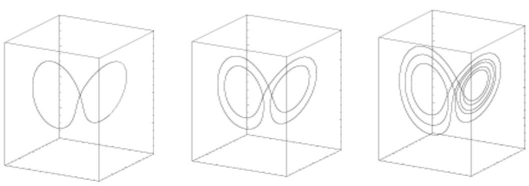


- exponential divergence of neighboring trajectories
- *often* fractal
- covered densely by trajectories
- contain an infinite number of "unstable periodic orbits"…



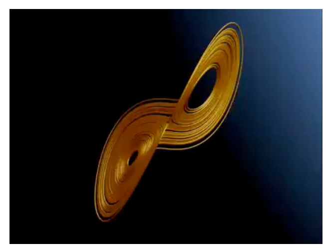
Unstable periodic orbits (UPOs):



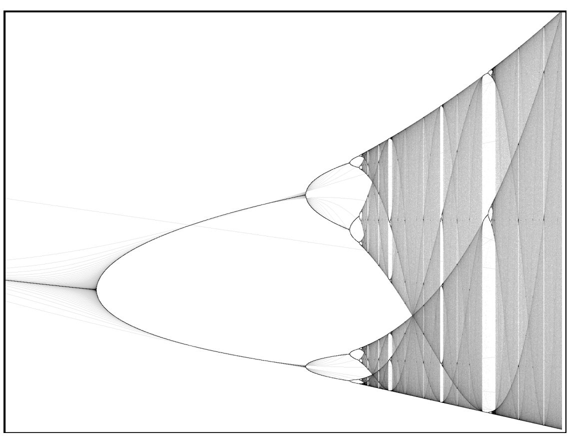


Bradley/Mantilla, *Chaos* **12**:596

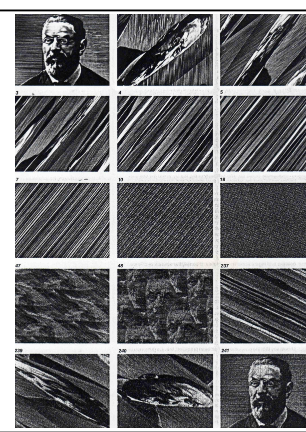
Attractor "bones"…



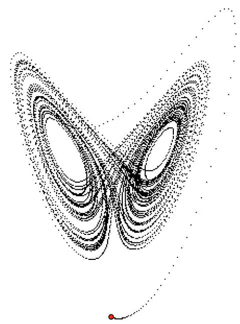
© 2006 Jos Leys and Etienne Ghys; `www.josleys.com`

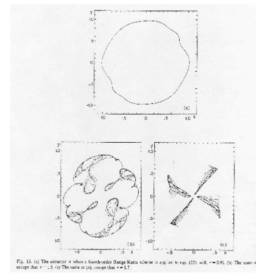


Poincare recurrence



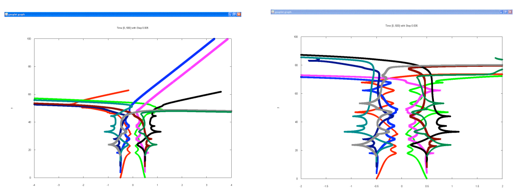
Crutchfield *et al*.
Chaos **255**:46





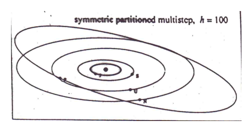
Different timestep

Lorenz, *Physica D* **35**:229

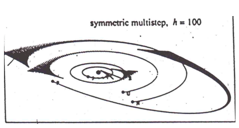


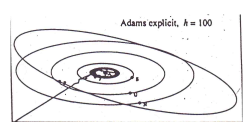
Different arithmetic

N. Ross Ph.D. thesis, Ucolorado, 2008



Different solver algorithm…





Moral: numerical methods can run amok in “interesting” ways…

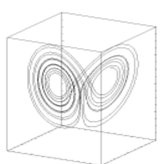
- can cause distortions, bifurcations, etc.
- and these look a lot like *real, physical* dynamics…
- source: algorithms, arithmetic system, timestep, etc.
- Q: what could you do to diagnose whether your results included spurious numerical dynamics?

Moral: numerical methods can run amok in “interesting” ways…

- can cause distortions, bifurcations, etc.
- and these look a lot like *real, physical* dynamics…
- source: algorithms, arithmetic system, timestep, etc.
- Q: what could you do to diagnose whether your results included spurious numerical dynamics?
    - *change the timestep*
    - *change the method*
    - *change the arithmetic*

So ODE solvers make mistakes.

…and chaotic systems are sensitively dependent on initial conditions….



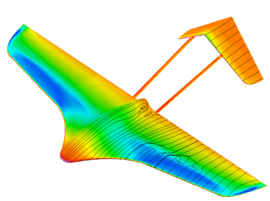
…??!?

Shadowing lemma:

Every noise-added trajectory on a chaotic attractor is *shadowed* by a true trajectory.

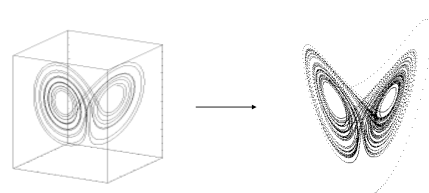
Important: this is for *state* noise, not *parameter* noise.

Solving *PDEs*

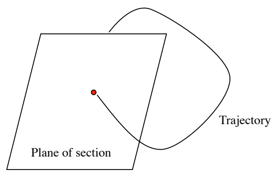


www.tecplot.com

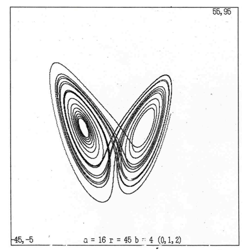
Projection:

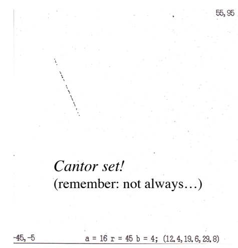


Section:



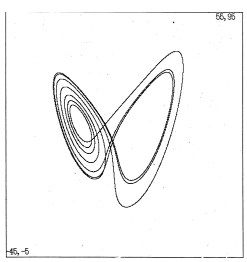
Section of a chaotic attractor:



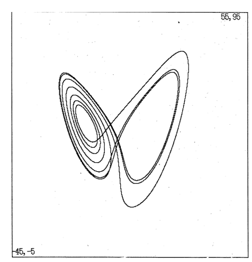


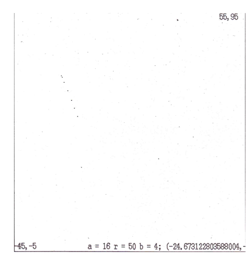
Cantor set!
(remember: not always…)

Section of a UPO:

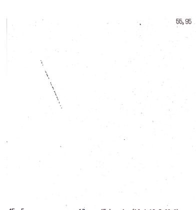


?





Aside: finding UPOs



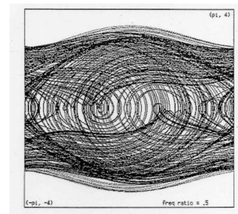
- Section
- Look for close returns
- Cluster
- Average
- See Gunaratne, So papers

Back to sections…*time-slice* ones now.

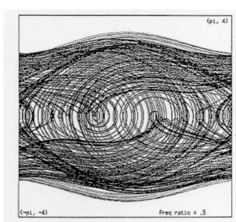
Time-slice sections of periodic orbits: some thought experiments

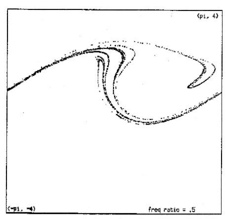
- pendulum rotating @ 1 Hz and strobe @ 1 Hz?
- pendulum rotating @ 1 Hz and strobe @ 2 Hz?
- pendulum rotating @ 1 Hz and strobe @ 3 Hz?
- pendulum rotating @ 1 Hz and strobe @ 1/2 Hz?
- pendulum rotating @ 1 Hz and strobe @ π Hz? (or some other irrational)

When this becomes really useful:

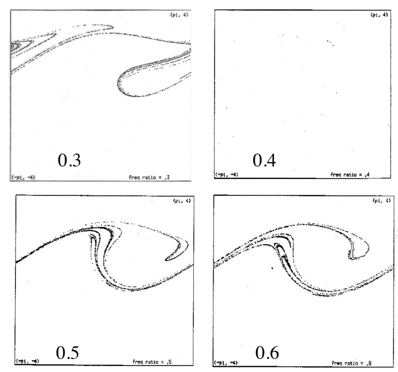


Poincare section:





What bifurcations look like on a Poincare section:



Computing sections:

- Space-slice
- Time-slice

Stability, λ, and the un/stable manifolds

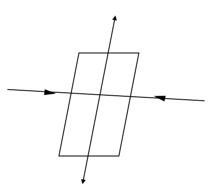
Lyapunov exponents:

- nonlinear analogs of eigenvalues: one λ for each dimension
- Σλ < 0 for dissipative systems
- λ are same for all ICs in one basin
- negative λ compress state space along *stable manifolds*
- positive λ stretch it along *unstable manifolds*
- biggest one λ_1 dominates as t → infinity
- *positive λ_1 is a signature of chaos*

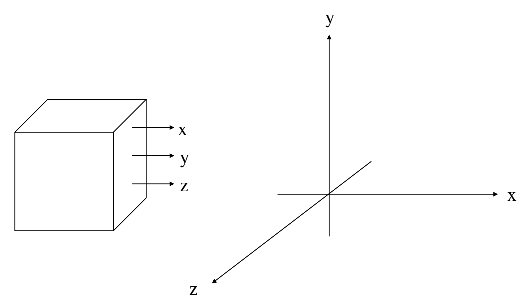
Calculating λ (& other invariants) *from data*

- A good reference: *Nonlinear Time Series Analysis* (Abarbanel's book is also very good)
- Associated software: TISEAN `www.mpipks-dresden.mpg.de/~tisean`
- *Be careful! TISEAN has lots of knobs and its results are incredibly sensitive to their values!*
- Use your dynamics knowledge to understand & use those knobs intelligently

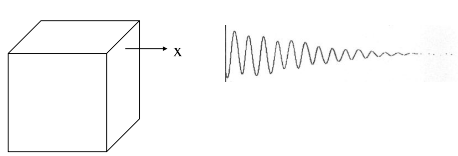
These λ & manifolds play a role in control of chaos…



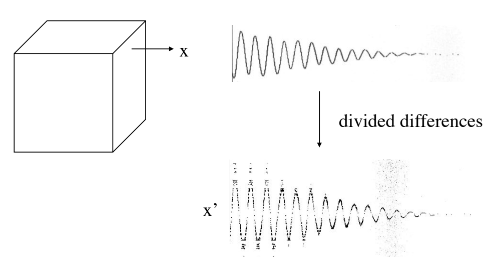
We've been assuming that we can measure all the state variables:



But often you can't:

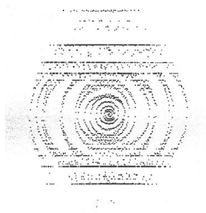


How to reconstruct the other state vars?



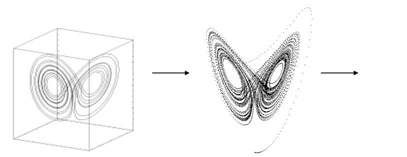
derivatives magnify noise!

What this looks like in the state space:



This is not useful for computation.

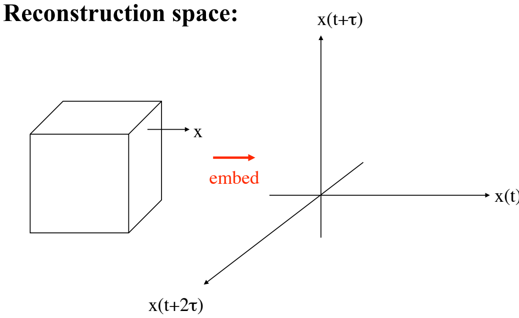
What we want here is to undo a projection:



Delay-coordinate embedding:

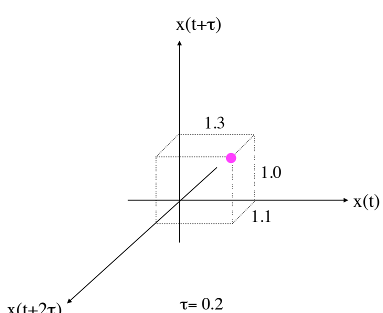
"reinflate" that squashed data to get a *topologically identical* copy of the original thing.

Reconstruction space:



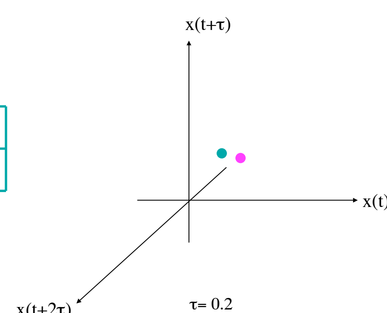
Mechanics:

x	t
1.3	0.1
1.2	0.2
1.0	0.3
0.8	0.4
1.1	0.5
1.4	0.6
1.6	0.7



$\tau = 0.2$
$m = 3$

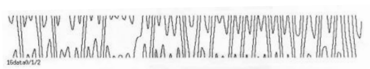
x	t
1.3	0.1
1.2	0.2
1.0	0.3
0.8	0.4
1.1	0.5
1.4	0.6
1.6	0.7



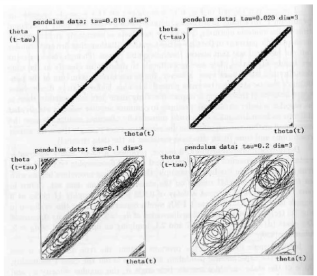
$\tau = 0.2$
$m = 3$

TISEAN does this

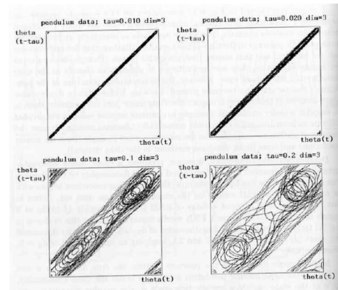
What this looks like:

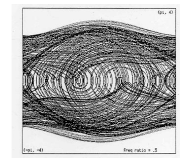
Data: 

Reconstruction space:



Bradley, in *Intelligent Data Analysis*, Springer 2000.





Bradley, in *Intelligent Data Analysis*, Springer 2000.

Takens* theorem:

For the right τ and enough dimensions, the dynamics in this *reconstruction space* are diffeomorphic to the original state-space dynamics.

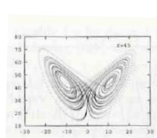
* Whitney, Mañé, …

Diffeomorphisms and topology:

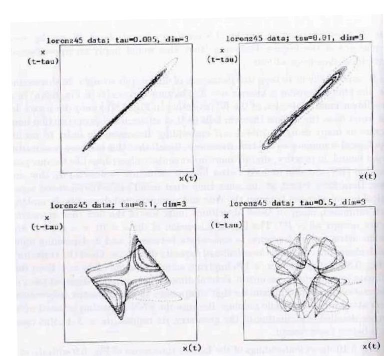
Diffeomorphic: mapping from the one to the other is differentiable and has a differentiable inverse.

What that means:

- *qualitatively* the same shape
- have same dynamical invariants (e.g., λ)

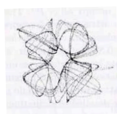


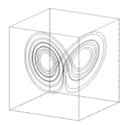
(b)



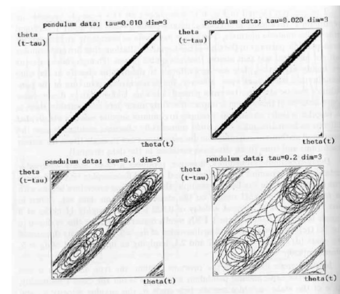
Takens* theorem:

For the right τ and enough dimensions, the embedded dynamics are diffeomorphic to (have same topology as) the original state-space dynamics.





Picking τ:



TISEAN contains tools that help you do this (e.g., `mutual`)

Picking *m*:

m > 2d: sufficient to ensure no crossings in reconstruction space:

…may be overkill.

"Embedology" paper: *m* > 2 dc
(box-counting dimension)

TISEAN contains tools that help you do this (e.g., `false_nearest`)

If Δt is not uniform:

~~Theorem (Takens): for τ>0 and m > 2d, reconstructed trajectory is diffeomorphic to the true trajectory~~

~~Conditions: evenly sampled in time~~

Interspike interval embedding:

idea: lots of systems generate spikes — hearts, nerves, etc.

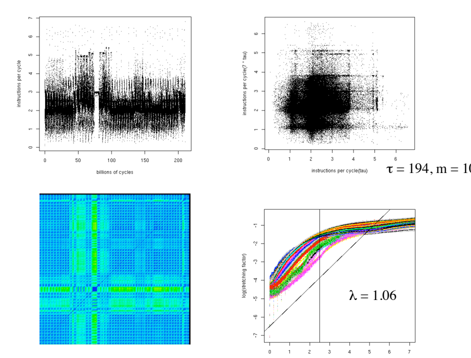
if you assume that the spikes are the result of an integrate-and-fire system, then the Δt has a one-to-one correspondence to some state variable's *integrated* value…

in which case the Takens theorem still holds.

(with the Δts as state variables)

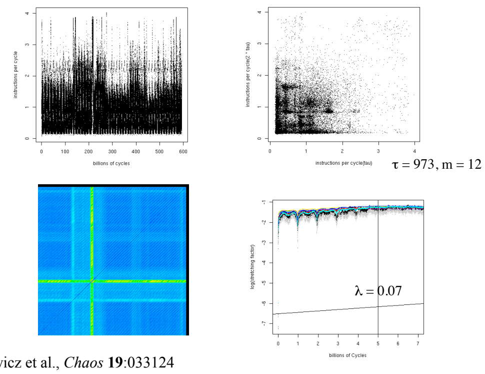
Sauer *Chaos* **5**:127

`bzip2` dynamics on an Intel Core2



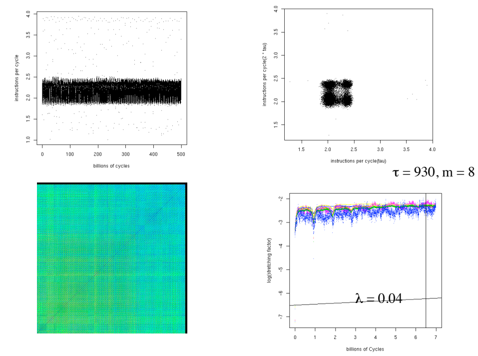
Mytkowicz et al., *Chaos* **19**:033124

`bzip2` dynamics on an Intel Pentium 4



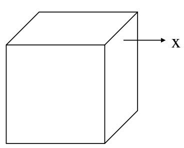
Mytkowicz et al., *Chaos* **19**:033124

`povray` dynamics on an Intel Core2

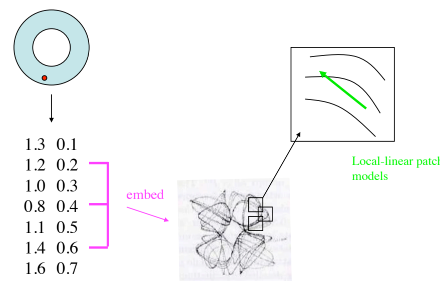


Mytkowicz et al., *Chaos* **19**:033124

What if the black-box system were a roulette wheel?

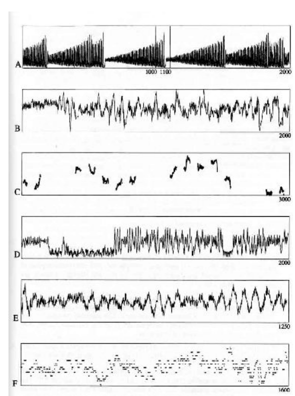


The Eudaemonic Pie
(or The Newtonian Casino)



The Santa Fe competition:

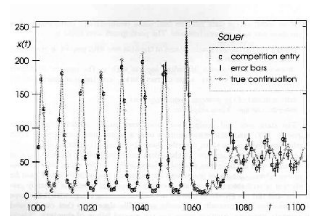
- Weigend & Gershenfeld, 1992
- put a bunch of data sets up on an ftp server
- and invited all comers to predict their future
- chronicled in *Time Series Prediction: Forecasting the Future and Understanding the Past*, Santa Fe Institute, 1993 (from which the images on the following half-dozen slides were reproduced)



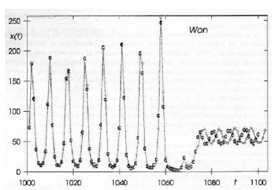
The Santa Fe competition: data

- Laboratory laser
- Medical data (sleep apnea)
- Currency rate exchange
- RK4 on some chaotic ODE
- Intensity of some star
- A Bach fugue

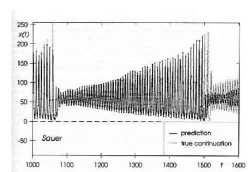
Embedding + patch models: (Sauer)



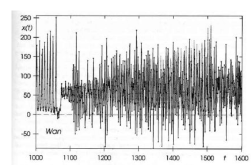
Neural net: (Wan)



Further out:

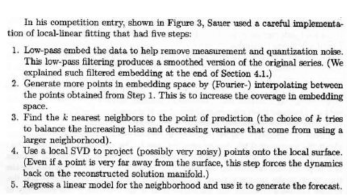


Sauer



Wan

Sauer’s algorithm:



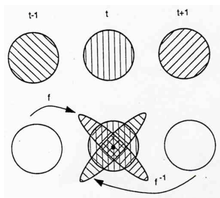
In his competition entry, shown in Figure 3, Sauer used a careful implementation of local-linear fitting that had five steps:

1. Low-pass embed the data to help remove measurement and quantization noise. This low-pass filtering produces a smoothed version of the original series. (We explained such filtered embedding at the end of Section 4.1.)
2. Generate more points in embedding space by (Fourier-) interpolating between the points obtained from Step 1. This is to increase the coverage in embedding space.
3. Find the k nearest neighbors to the point of prediction (the choice of k tries to balance the increasing bias and decreasing variance that come from using a larger neighborhood).
4. Use a local SVD to project (possibly very noisy) points onto the local surface. (Even if a point is very far away from the surface, this step forces the dynamics back on the reconstructed solution manifold.)
5. Regress a linear model for the neighborhood and use it to generate the forecast.

Filtering:

Linear: a bad idea if the system is chaotic

Nonlinear: use the stable and unstable manifold structure on a chaotic attractor…

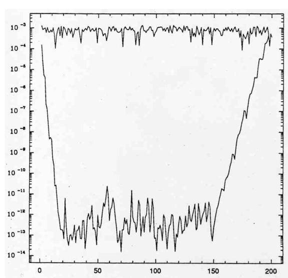


Farmer & Sidorowich, in *Evolution, Learning and Cognition*, World Scientific, 1983

Idea:

- If you have a model of the system, you can simulate what happens to each point in forward *and backward* time
- If your system has transverse stable and unstable manifolds, that does useful things to the noise balls
- Since all three versions of that data should be identical at the middle time, can average them
- ⇨ noise reduction!
- Works best if manifolds are perpendicular, but requires only transversality

Results:



Farmer & Sidorowich, in *Evolution, Learning and Cognition*, World Scientific, 1983

Computational Topology:

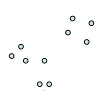
Why: this is the fundamental mathematics of shape. complements geometry.

What: compute topological properties *from finite data*

How:
• introduce resolution parameter
• count components and holes at different resolutions
• deduce topology from patterns therein

Connectedness: definitions

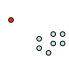
• how many "lumps" in a data set:



• ε-connectedness (after Cantor)

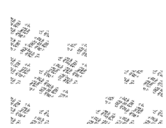
• ε-connected components

• ε-isolated points:

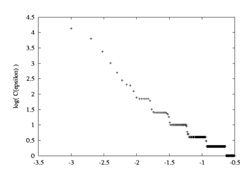


Connectedness: examples

If the data points are samples of a disconnected fractal like this:



The number of connected components looks like this:

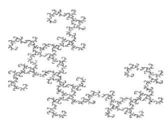


(note obvious tie-in to fractal dimension…)

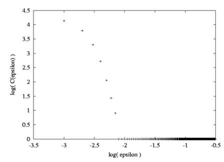
Robins *et al.*, *Physica D* **139**:276, *Nonlinearity* **11**:913

Connectedness: examples

If the data points are samples of a connected set like this:



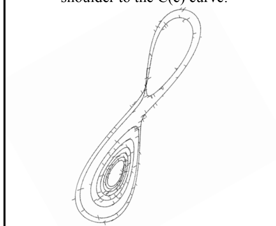
The number of connected components looks like this:

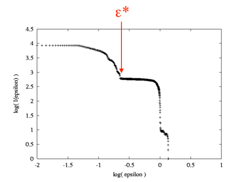


Robins *et al.*, *Physica D* **139**:276, *Nonlinearity* **11**:913

Connectedness: applications

The effects of noise are to add isolated points to the set and a shoulder to the C(e) curve:

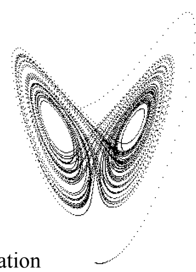




So if you know that the object is connected, you can reasonably assume that any isolated points are noisy, and remove them by pruning with ε = ε*

Robins *et al.*, *Intelligent Data Analysis* **8**:505, *Chaos* **14**:305

Chaos and control…

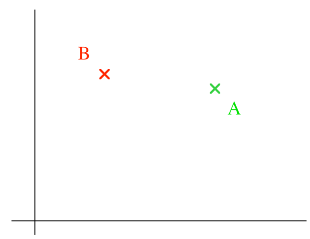


key concepts:

- dense attractor coverage
- exponential trajectory separation
- un/stable manifold structure
- local-linear control

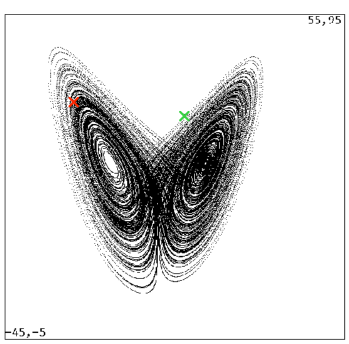
Control:

getting from A to B, minimizing some cost functional…



Lorenz System:

denseness, reachability, and control



R = 50

OGY control: taking advantage of the unique properties of chaos…

- dense attractor coverage → reachability (*)
- un/stable manifold structure → controllability

* *eventually…*

Ott *et al.*, *PRL* **64**:1196

- dense attractor coverage → reachability
- un/stable manifold structure → controllability
- exploit sensitive dependence, too???

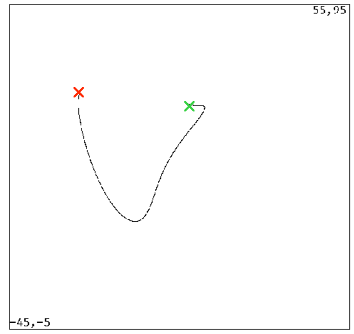
⟹ "targeting"

Lorenz System:

SDOIC-based targeting

OGY & co. have been used in *tons* of systems; see Shinbrot review paper.

Also see Hubler's work on open-loop control.



Four R switches; 240X faster

Bradley, *Cybernetics & Systems* **26**:299

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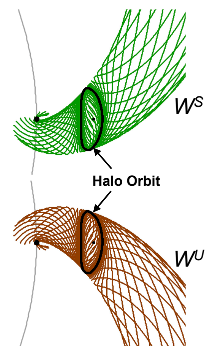
(303) 492-4668

Using invariant manifolds…

Want to get a spacecraft onto a "halo orbit," which is a UPO of the dynamics.

Unstable Periodic Orbits (UPOs) have invariant manifolds:

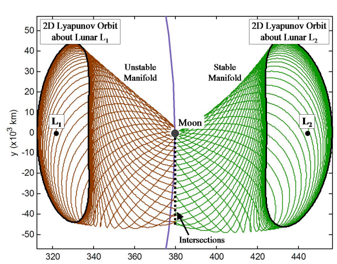
- Stable Invariant Manifold (W^S)
  - *The set of all trajectories a particle could use to arrive onto the UPO.*
- Unstable Invariant Manifold (W^U)
  - *The set of all trajectories a particle could take after a small perturbation from the UPO.*

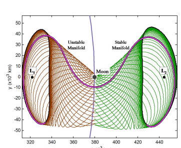


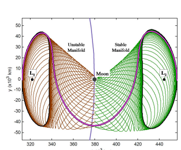
Jeff Parker, PhD thesis, UColorado 2008

Low-energy (cheap) orbit transfers

- Depart along W^U_{L1} & arrive on W^S_{L2}



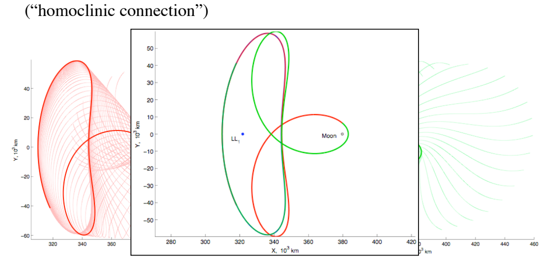




Jeff Parker, PhD thesis, UColorado 2008

Homoclinic orbits - The best case

- If a trajectory in Stable and Unstable intersect ("homoclinic connection")



Unstable Manifold of an LL_1 Lyapunov Orbit

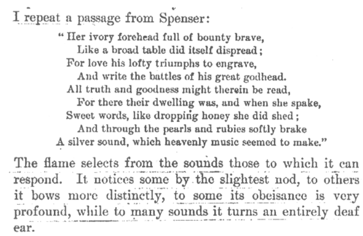
Stable Manifold of an LL_1 Lyapunov Orbit

Jeff Parker, PhD thesis, UColorado 2008

Can we do any of that in spatially extended systems?

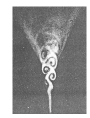
(i.e. harness the butterfly effect, exploit un/stable manifold geometry?)

- Sensitive flames (1856 – 1930's)

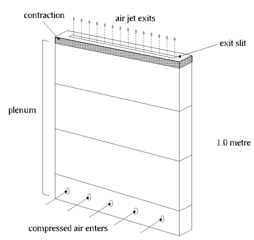

I repeat a passage from Spenser:

"Her ivory forehead full of bounty brave,
Like a broad table did itself dispread;
For love his lofty triumphs to engrave,
And write the battles of his great godhead.
All truth and goodness might therein be read,
For there their dwelling was, and when she spake,
Sweet words, like dropping honey she did shed;
And through the pearls and rubies softly brake
A silver sound, which heavenly music seemed to make."

The flame selects from the sounds those to which it can respond. It notices some by the slightest nod, to others it bows more distinctly, to some its obeisance is very profound, while to many sounds it turns an entirely deaf ear.

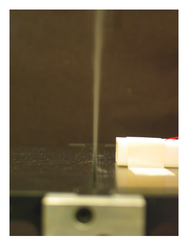


2D jet apparatus

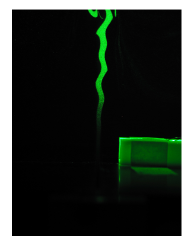


Peacock *et al*., *Exp. Fluids* **37**:22

End view



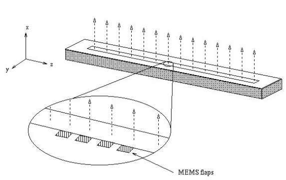
room lighting



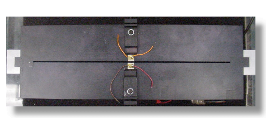
stop-action laser "slice"

aerosolized canola oil

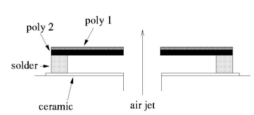
Forcing the jet flow

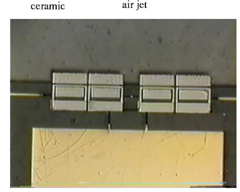


Slit: 2.5 X 400 mm



MEMS actuators





Video : overhead view at 2Hz, 10V

Area of individual flap is 1.0 x 0.25mm

Ma *et al*., *IEEE Trans. Adv. Packaging* **26**:268

The Butterfly effect in action…

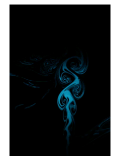
no forcing *6Hz forcing*

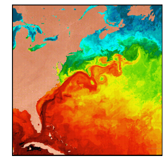
MEMS flap

Forcing generates coherent structures that enhance entrainment and mixing

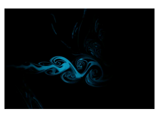
Peacock *et al.*, *Exp. Fluids* **37**:22

Does this have anything to do with reality?



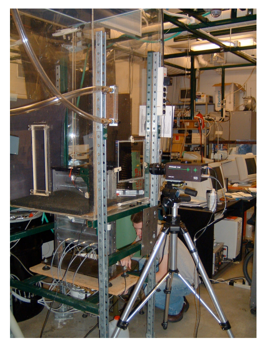


Flow control

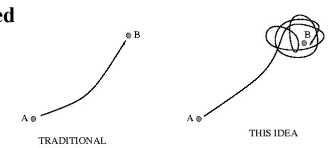


- Idea: co-opt system's natural instability to "grow" small perturbation into desired downstream effect
- Motivation: automotive, aerospace, combustion, …
- Challenge: control the convective instability of a spatiotemporally extended nonlinear system (ouch)

Measurement & Isolation:



"Chaos-enhanced reachability"



- **can control position/volume/density of attractor –*within limits***
- **possibly not reachable any other way**
- **nondeterministic – not for time-critical applications**

Using Chaos to Broaden the Capture Range of a Phase-Locked Loop

Elizabeth Bradley, *Member, IEEE*

Communication and chaos:

- Two coupled Lorenz systems will synchronize
- Robust w.r.t. a small amount of noise
- Use this to transmit & receive information

$$\begin{aligned} x' &= a(y-x) \\ y' &= rx - y - xz \\ z' &= xy - bz \end{aligned} \longrightarrow \begin{aligned} x' &= a(y-x) \\ y' &= r(x+\varepsilon x) - y - xz \\ z' &= xy - bz \end{aligned}$$

- Chaotic carrier wave, so hard to intercept or jam

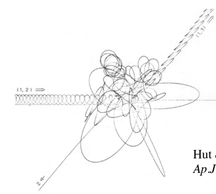
Pecora & Carroll *Phys. Rev. Lett* **64**:821

Another interesting application: chaos in the solar system

- orbits of Pluto, Mars
- Kirkwood gaps
- rotation of Hyperion & other satellites
- …

Solar system stability:

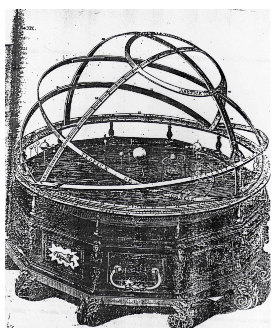
- recall: two-body problem not chaotic
- but three (or more) can be…



Hut & Bahcall
Ap.J. ***268**:319*

Exploring that issue, circa 1880:

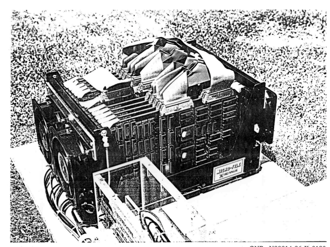
- an *orrery*



Exploring that issue, circa 1980:

- write the *n*-body equations for the solar system
- solve them on a special-purpose computer

The *digital orrery*
(Wisdom & Sussman)



ONR N00014-86-K-0180

Numerical Evidence That the Motion of Pluto Is Chaotic

Gerald Jay Sussman and Jack Wisdom

The Digital Orrery has been used to perform an integration of the motion of the outer planets for 845 million years. This integration indicates that the long-term motion of the planet Pluto is chaotic. Nearby trajectories diverge exponentially with an *e*-folding time of only about 20 million years.

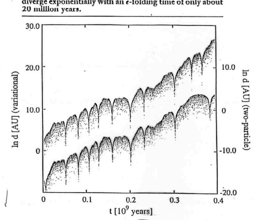


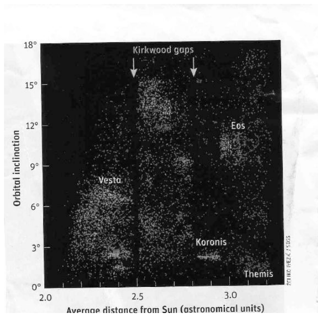
Figure 3: The exponential divergence of nearby trajectories is indicated by the average linear growth of the logarithms of the distance measures as a function of time. In the upper trace we see the growth of the variational distance around a reference trajectory. In the lower trace we see how two Plutos diverge with time. The distance saturates near 80AU when the Plutos are on opposite sides of the Sun. The variational method of studying neighboring trajectories does not have the problem of saturation. Note that the two methods are in excellent agreement until the two-trajectory method has nearly saturated.

Science **241**:433

Should we worry?

- No.

Kirkwood gaps:



From *Sky & Telescope*

Chaos and the Kirkwood gaps

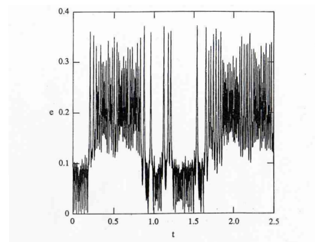
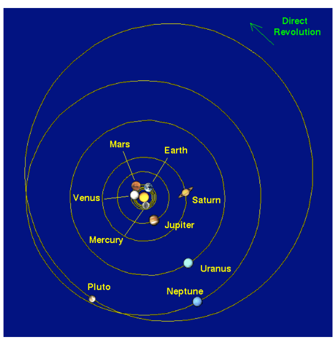


FIGURE 5. Eccentricity of a typical chaotic trajectory over a longer time interval. the time is now measured in millions of years. Bursts of high eccentricity behavior are interspersed with intervals of irregular low eccentricity behavior, broken by occasional spikes.

Wisdom, *Nuclear Phys. B* **2**:391



csep10.phys.utk.edu/astr161/lect/solarsys/revolution.html

Evidence in favor of the conjecture:

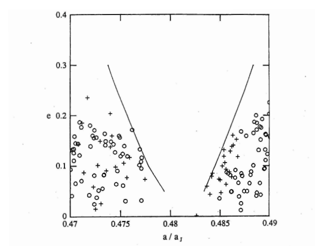
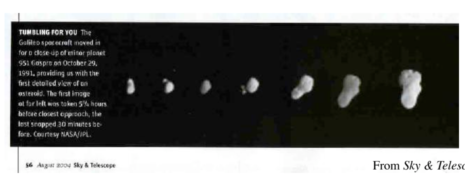


FIGURE 9. Comparison of the actual distribution of asteroids with the outer boundaries of the chaotic zone. There is both a chaotic region and quasiperiodic region in the gap, but trajectories of both types are planet crossing.

Wisdom, *Nuclear Phys. B* **2**:391

Chaotic tumbling of satellites:

Voyager and Galileo saw this…



From *Sky & Telescope*

Ap. J. **97**:570
Ap. J. **98**:1855

www.nasa.gov/mission_pages/cassini/multimedia/pia06243.html

Chaotic tumbling of satellites:

This happens for **all** satellites at some point in their history, unless they are perfectly spherical and in perfectly circular orbits (pf: KAM theorem; see [53] on syllabus)

Some of them are still tumbling chaotically because of their geometry, but most (like the earth and its moon) have settled down into equilibria

More chaos in the solar system:

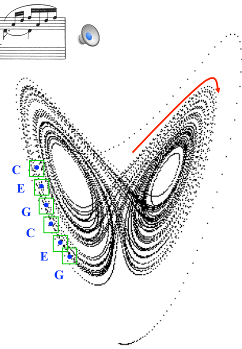
- obliquity of Mars (Touma & Wisdom, *Science* **259**:1294)

www.solarviews.com

- etc.

Musical Variations from a Chaotic Mapping

Pitch sequence:
C, E, G, C, E, G, C, E…



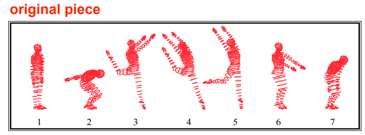
C **symbol dynamics**

variation!

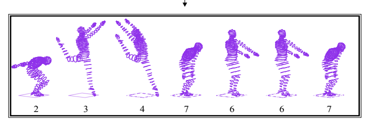
Dabby *Chaos* **6**:95

Chaotic variations on movement sequences

original piece

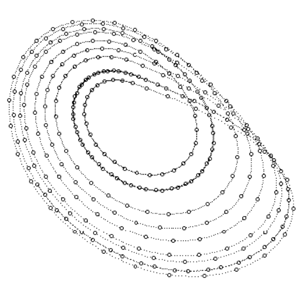


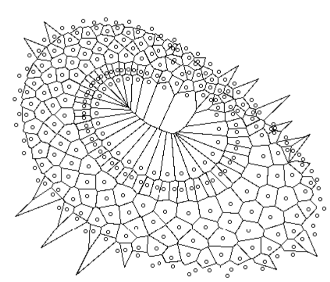
chaotic **mapping**

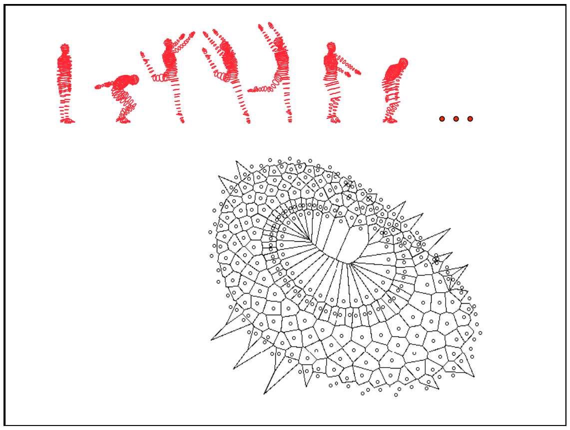


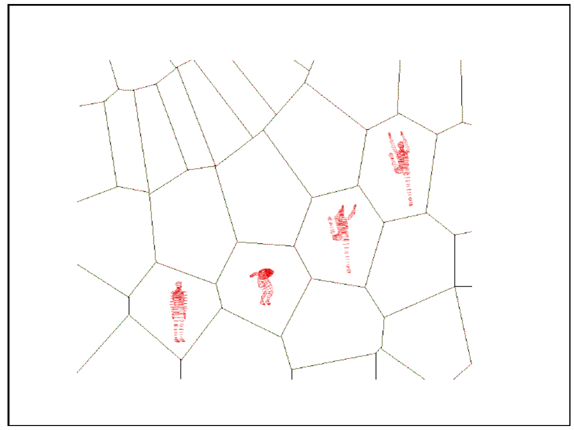
chaotic variation

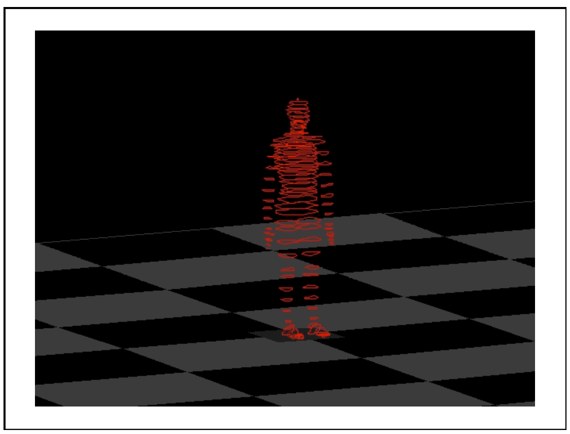
Bradley & Stuart, *Chaos* **8**:800

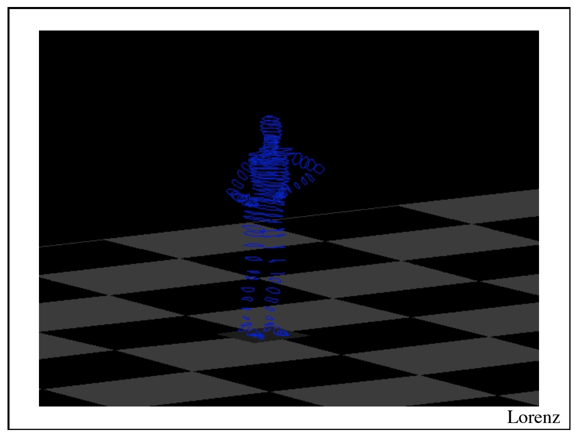


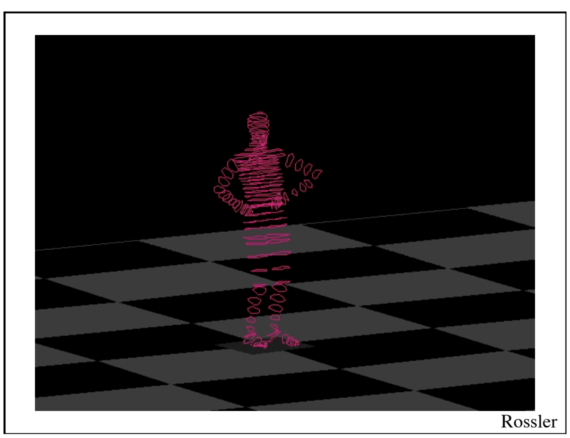


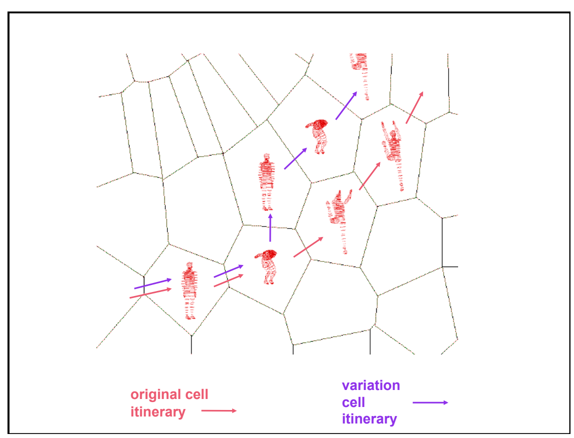


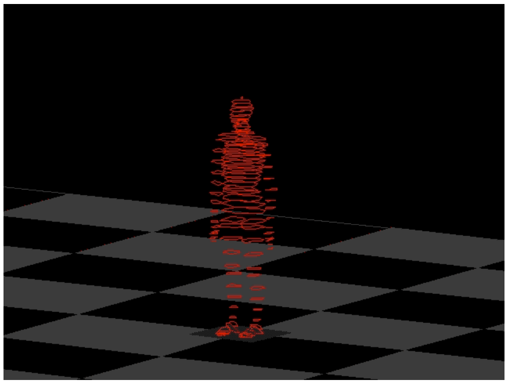




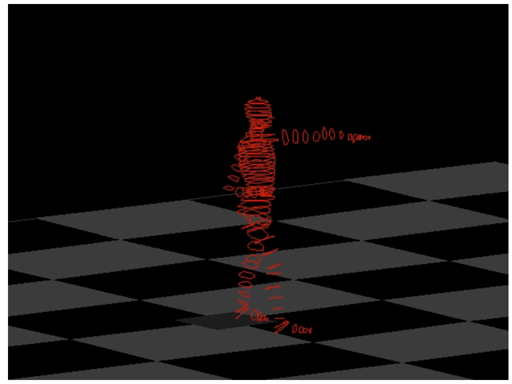

Lorenz


Rossler

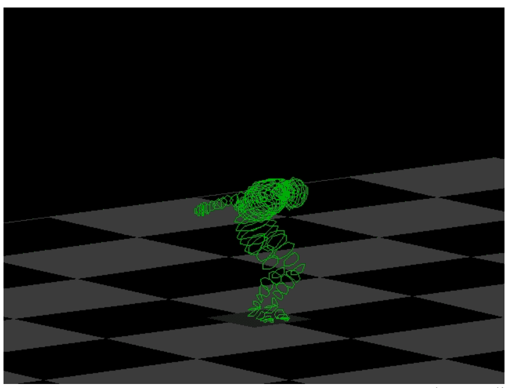

original cell
itinerary
variation
cell
itinerary



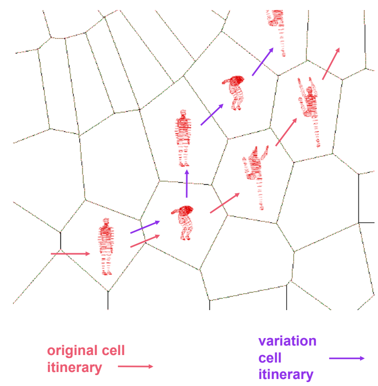
medley

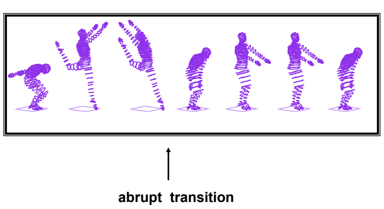


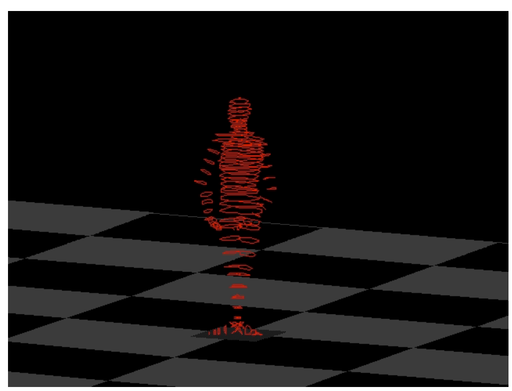
Rossler medley

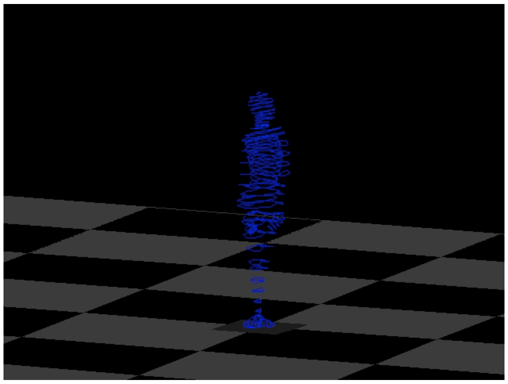


random medley

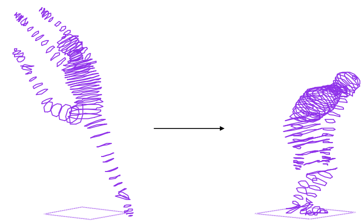






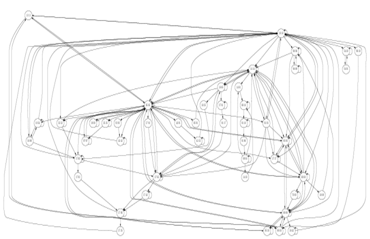


Interpolation



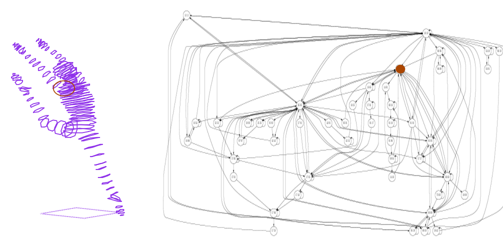
Stuart & Bradley, *ICML* 1998

Corpus-based approach

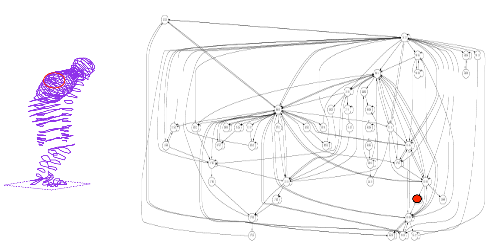


- graph captures motions of one joint
- note: specific to the genre of the corpus!

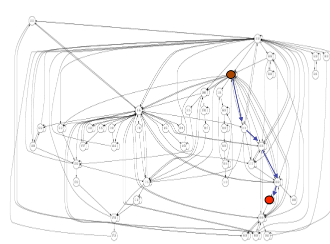
Initial state



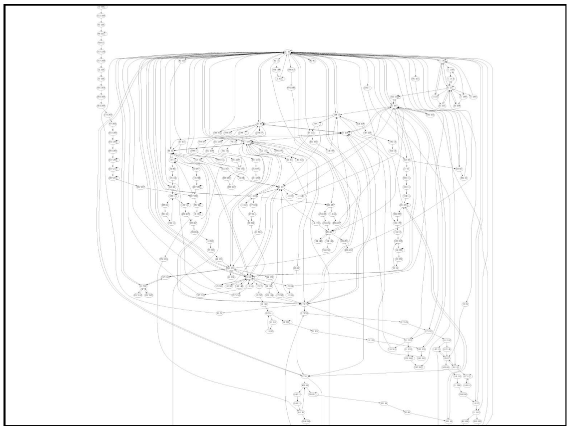
Target state

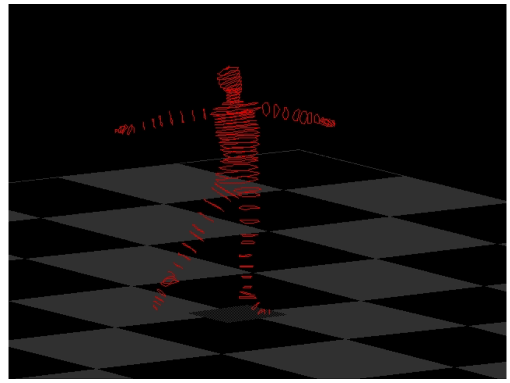


Graph search

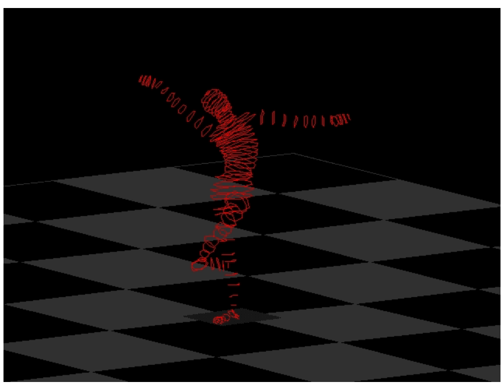


…for 44 joints in parallel!

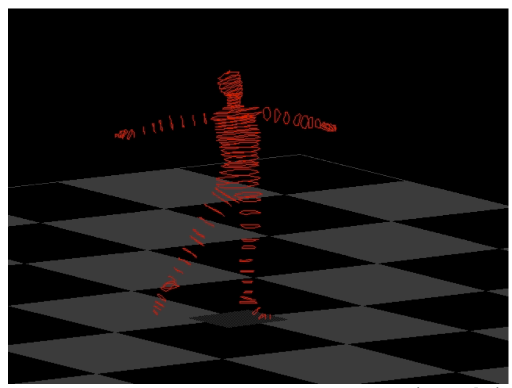




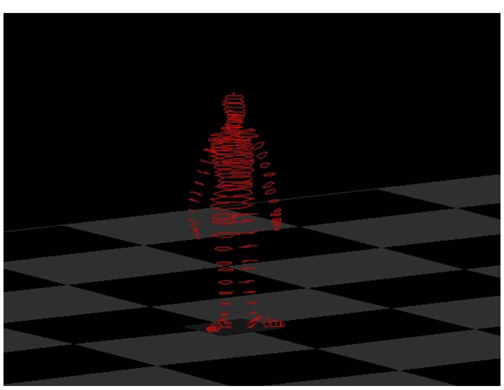
initial



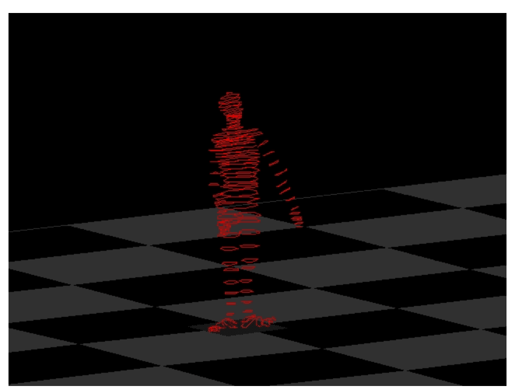
target



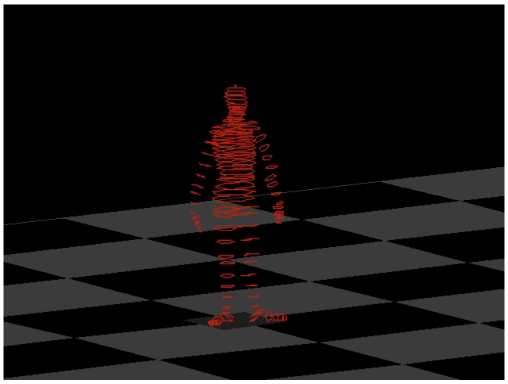
interpolation 1



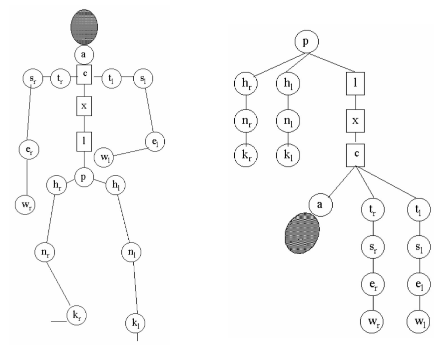
initial

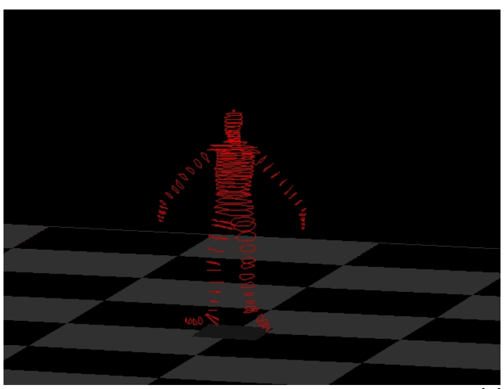


target

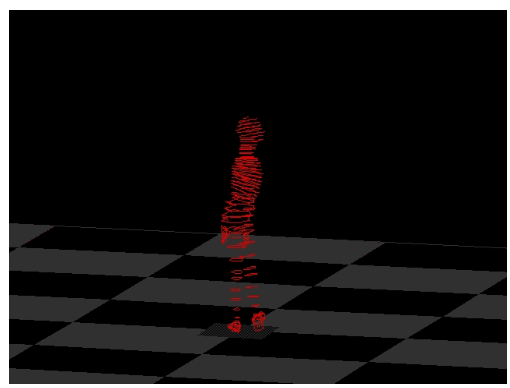


interpolation 2

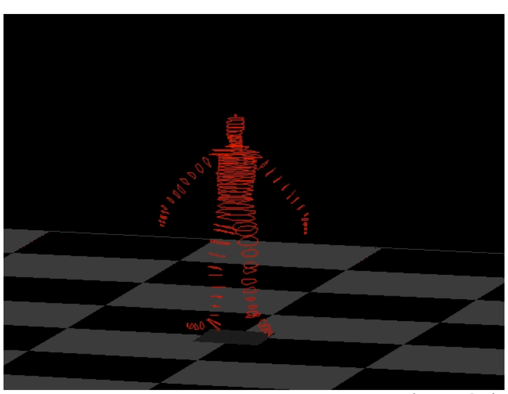




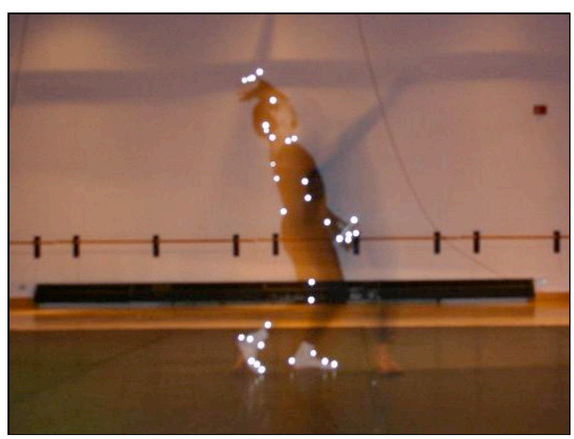
initial

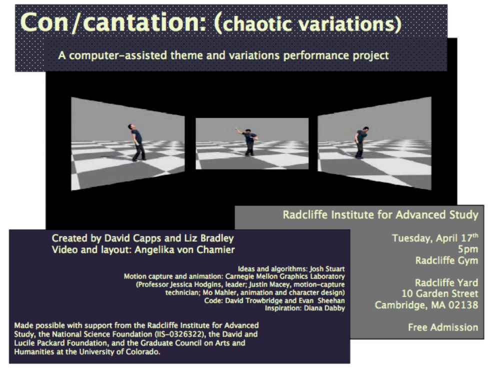


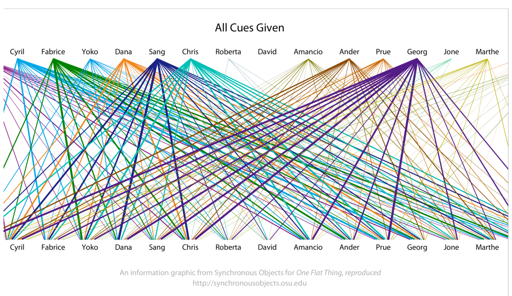
target



interpolation 3







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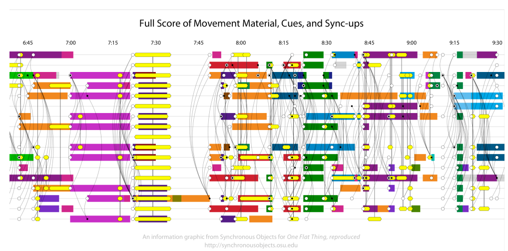

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