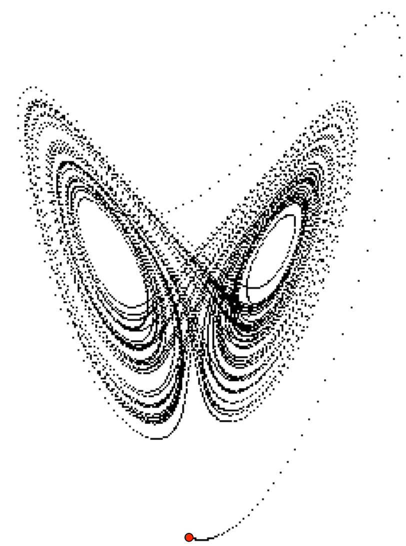


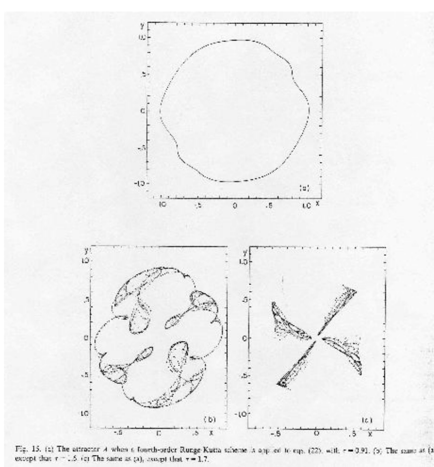
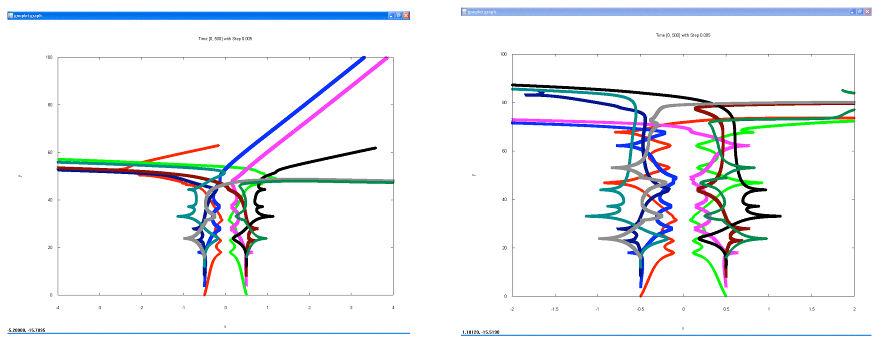


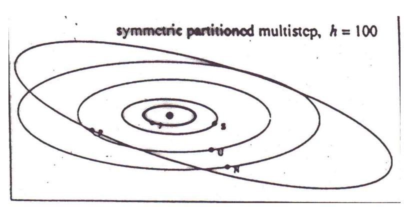
Fig. 15. (a) The attractor A when a fourth-order Runge-Kutta scheme is applied to eqs. (22), with $\tau = 0.91$. (b) The same as (a) except that $\tau = 1.5$. (c) The same as (a), except that $\tau = 1.7$.

Different timestep

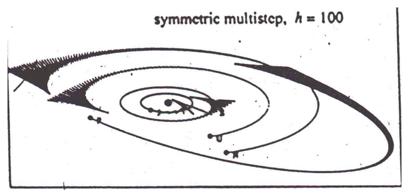
Lorenz, *Physica D* **35:**229

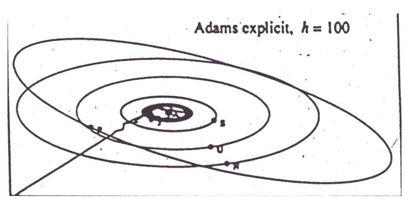


Different arithmetic



Different solver algorithm…


symmetric multistep, $h = 100$


Adams explicit, $h = 100$

Moral: numerical methods can run amok in "interesting" ways…

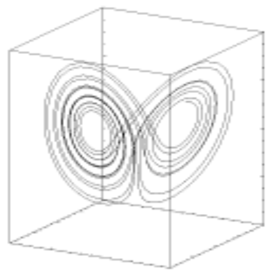
- can cause distortions, bifurcations, etc.
- and these look a lot like *real, physical* dynamics…
- source: algorithms, arithmetic system, timestep, etc.
- Q: what could you do to diagnose whether your results included spurious numerical dynamics?

Moral: numerical methods can run amok in "interesting" ways…

- can cause distortions, bifurcations, etc.
- and these look a lot like *real, physical* dynamics…
- source: algorithms, arithmetic system, timestep, etc.
- Q: what could you do to diagnose whether your results included spurious numerical dynamics?
    - *change the timestep*
    - *change the method*
    - *change the arithmetic*

So ODE solvers make mistakes.

…and chaotic systems are sensitively dependent on initial conditions….



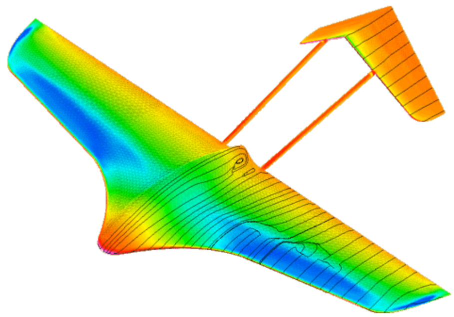
…??!?

Shadowing lemma:

Every noise-added trajectory on a chaotic attractor is *shadowed* by a true trajectory.

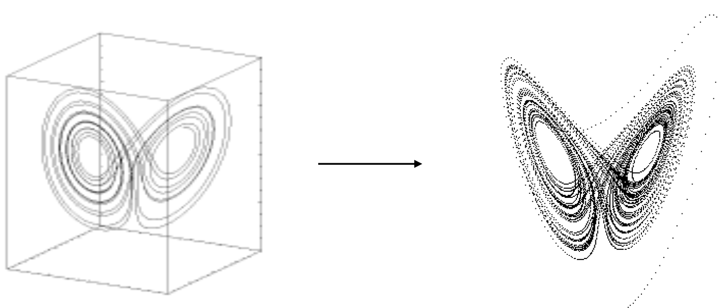
Important: this is for *state* noise, not *parameter* noise.

Solving *PDEs*

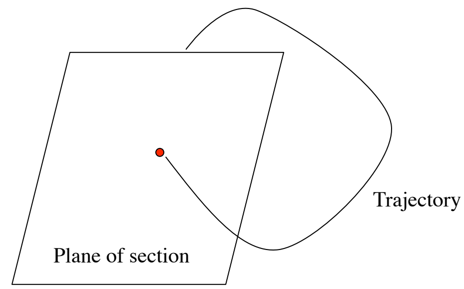


www.tecplot.com

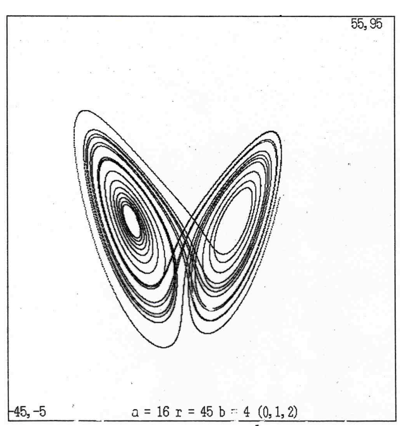
Projection:

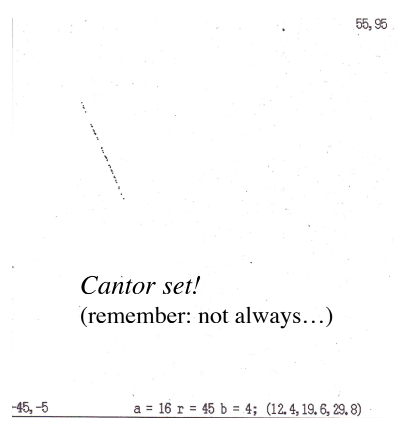


Section:



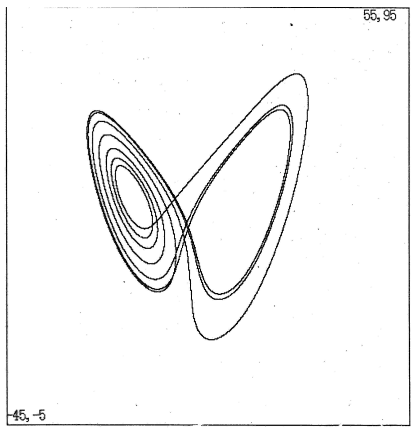
Section of a chaotic attractor:



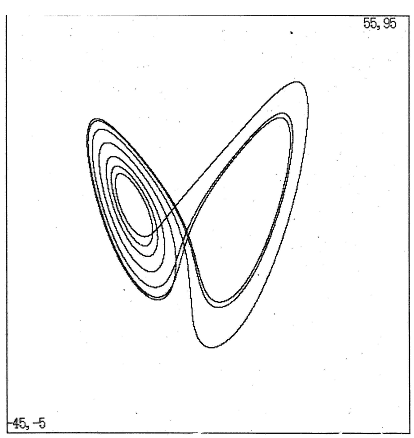


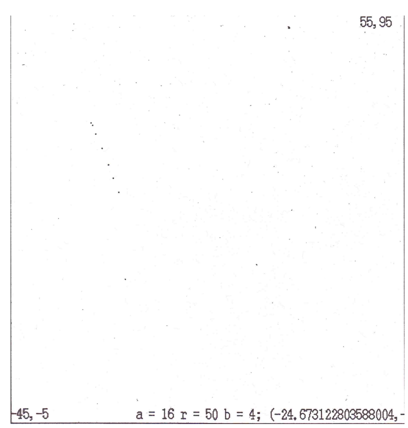
Cantor set!
(remember: not always…)

Section of a UPO:

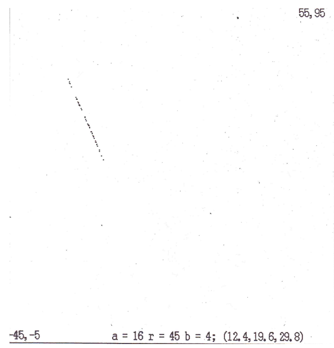


?





Aside: finding UPOs



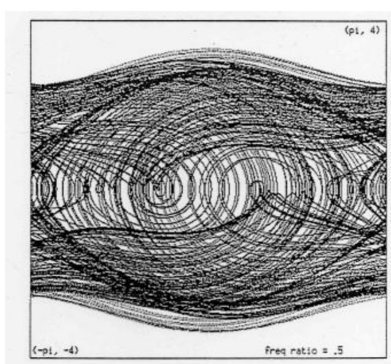
- Section
- Look for close returns
- Cluster
- Average
- See Gunaratne, So papers

Back to sections…*time-slice* ones now.

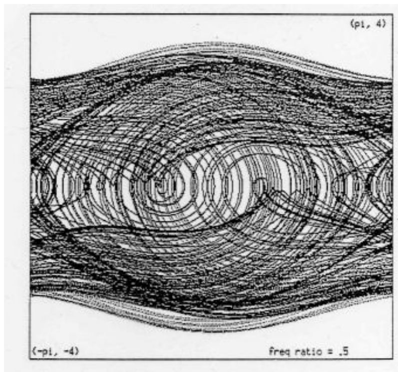
Time-slice sections of periodic orbits: some thought experiments

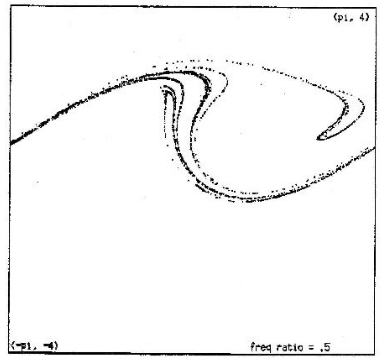
- pendulum rotating @ 1 Hz and strobe @ 1 Hz?
- pendulum rotating @ 1 Hz and strobe @ 2 Hz?
- pendulum rotating @ 1 Hz and strobe @ 3 Hz?
- pendulum rotating @ 1 Hz and strobe @ 1/2 Hz?
- pendulum rotating @ 1 Hz and strobe @ π Hz? (or some other irrational)

When this becomes really useful:

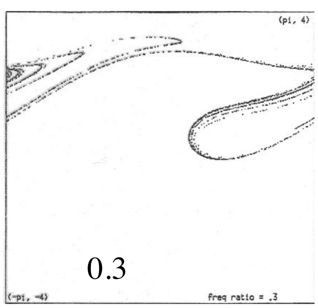


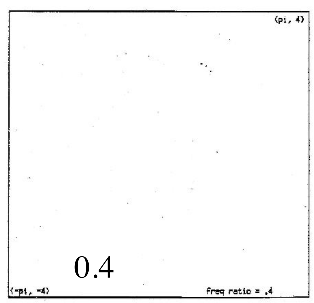
Poincare section:

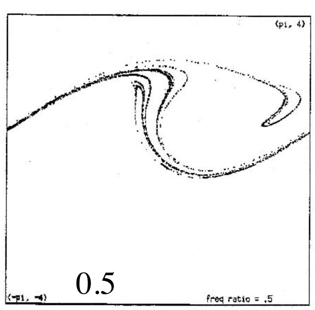


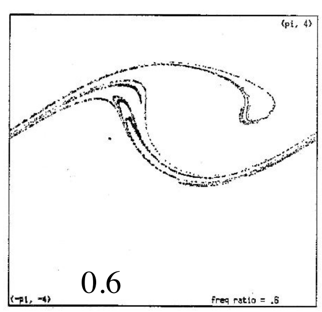


What bifurcations look like on a Poincare section:









Computing sections:

- Space-slice
- Time-slice

Stability, λ, and the un/stable manifolds

Lyapunov exponents:

- nonlinear analogs of eigenvalues: one λ for each dimension
- $\Sigma\lambda < 0$ for dissipative systems
- λ are same for all ICs in one basin
- negative λ compress state space along *stable manifolds*
- positive λ stretch it along *unstable manifolds*
- biggest one λ_1 dominates as t $\rightarrow$ infinity
- *positive λ_1 is a signature of chaos*

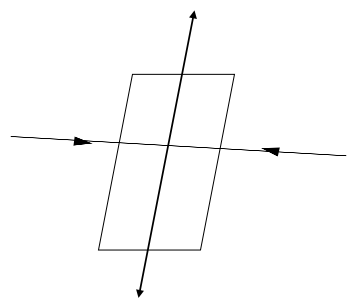
Calculating λ (& other invariants) *from data*

- "Nonlinear Time Series Analysis" (among others)
- TISEAN (url on syllabus)

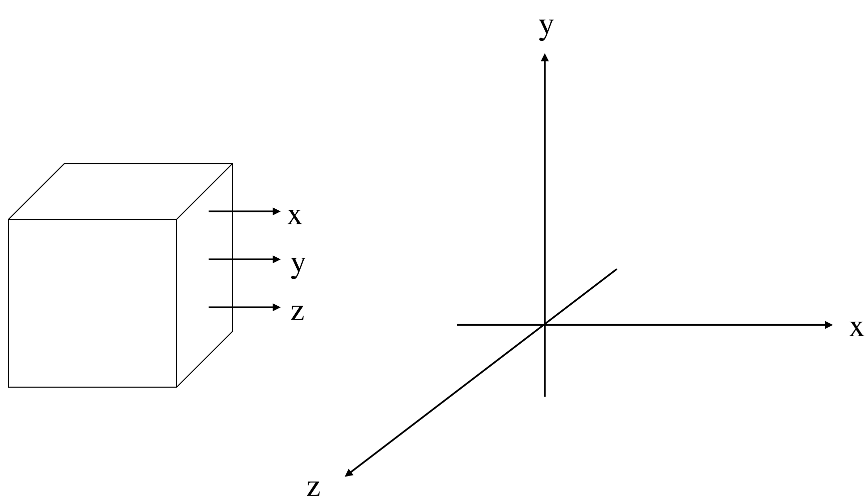
- *Be careful! This s/w has lots of knobs and its results are incredibly sensitive to their values!*

- Use your dynamics knowledge to understand & use those knobs intelligently

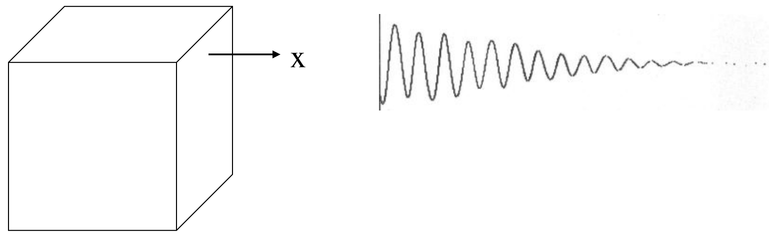
These λ & manifolds play a role in control of chaos…



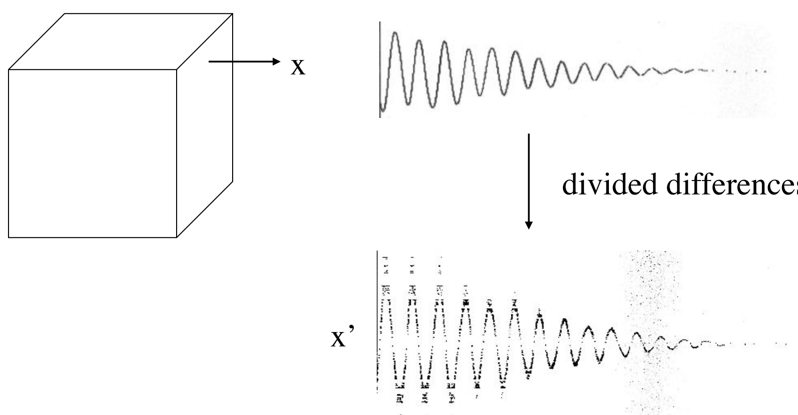
We've been assuming that we can measure all the state variables:



But often you can't:

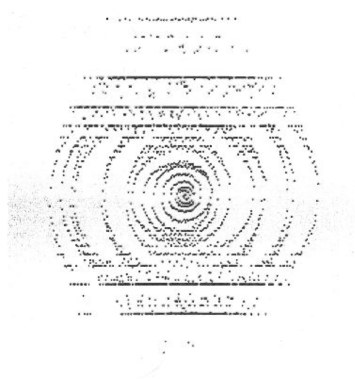


How to reconstruct the other state vars?



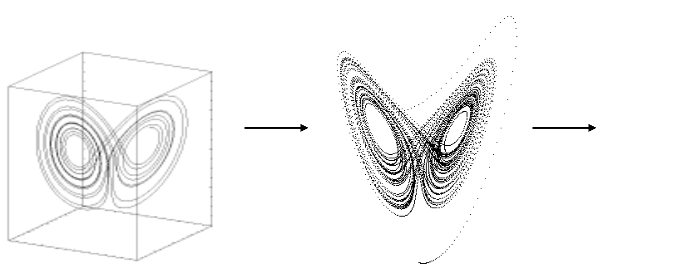
derivatives magnify noise!

What this looks like in the state space:



This is not useful for computation.

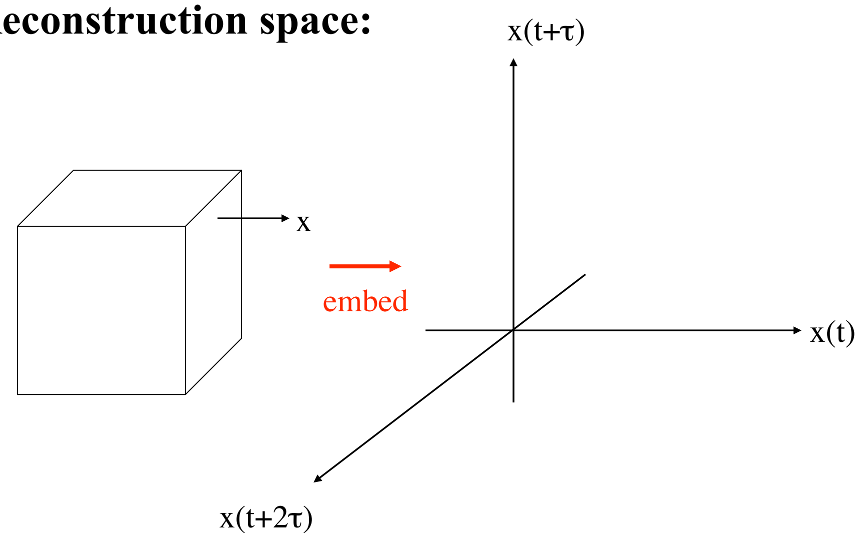
What we want here is to undo a projection:



Delay-coordinate embedding:

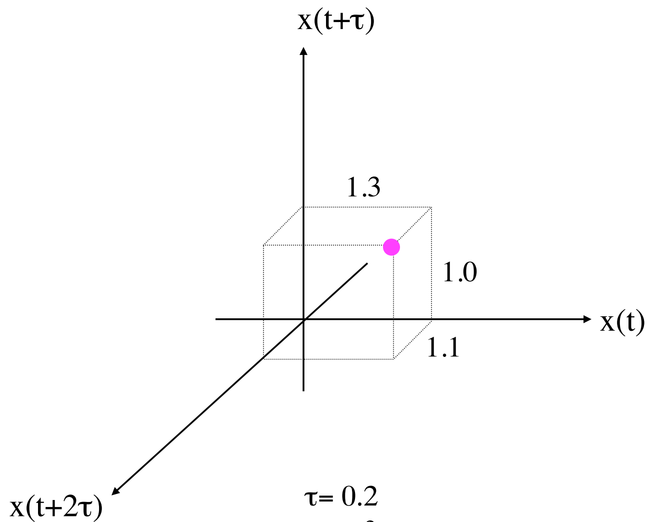
"reinflate" that squashed data to get a *topologically identical* copy of the original thing.

Reconstruction space:



Mechanics:

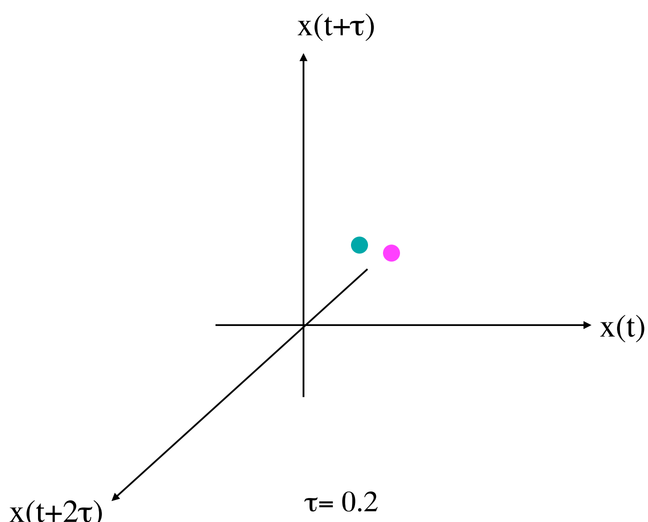
x	t
1.3	0.1
1.2	0.2
1.0	0.3
0.8	0.4
1.1	0.5
1.4	0.6
1.6	0.7



$\tau = 0.2$

$m = 3$

x	t
1.3	0.1
1.2	0.2
1.0	0.3
0.8	0.4
1.1	0.5
1.4	0.6
1.6	0.7



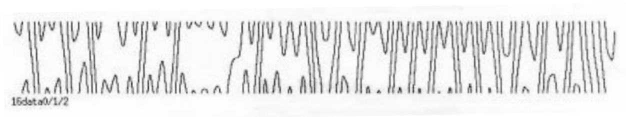
$\tau = 0.2$

$m = 3$

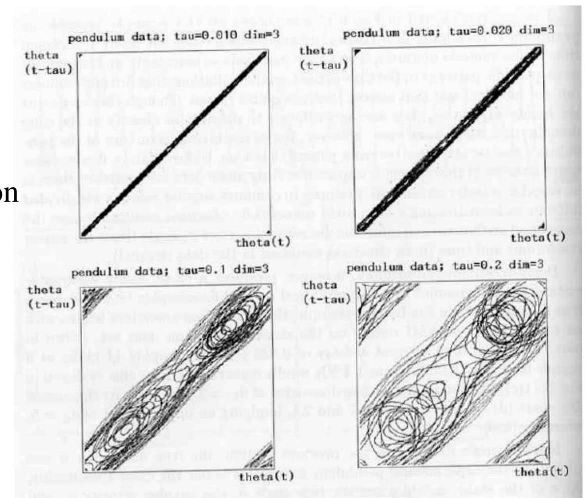
TISEAN does this

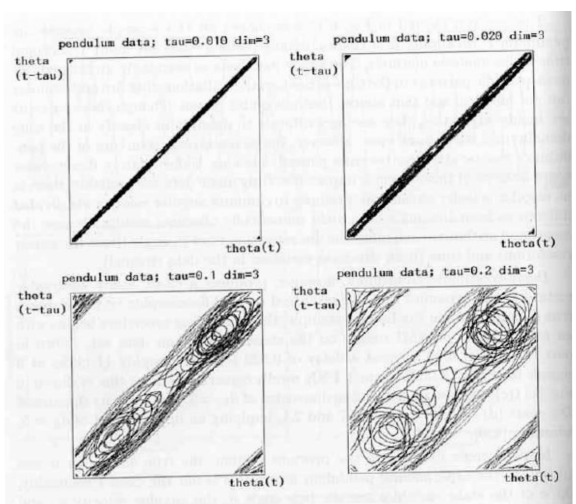
What this looks like:

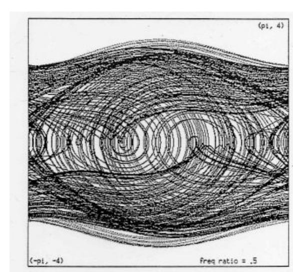
Data:



Reconstruction space:







Takens* theorem:

For the right τ and enough dimensions, the dynamics in this *reconstruction space* are diffeomorphic to the original state-space dynamics.

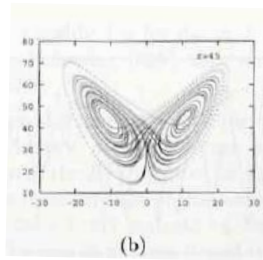
* Whitney, Mane, …

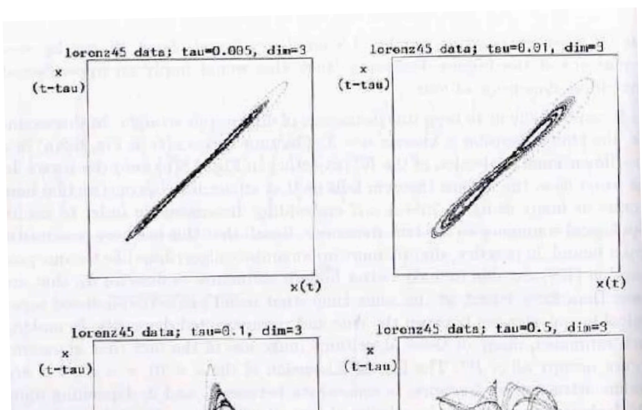
Diffeomorphisms and topology:

Diffeomorphic: mapping from the one to the other is differentiable and has a differentiable inverse.

What that means:

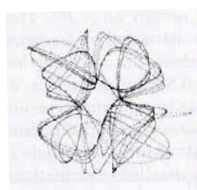
- *qualitatively* the same shape
- have same dynamical invariants (e.g., λ)

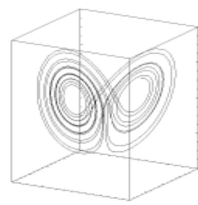

r=45
(b)


lorenz45 data; tau=0.005, dim=3
lorenz45 data; tau=0.01, dim=3
lorenz45 data; tau=0.1, dim=3
lorenz45 data; tau=0.5, dim=3
x (t-tau)
x(t)

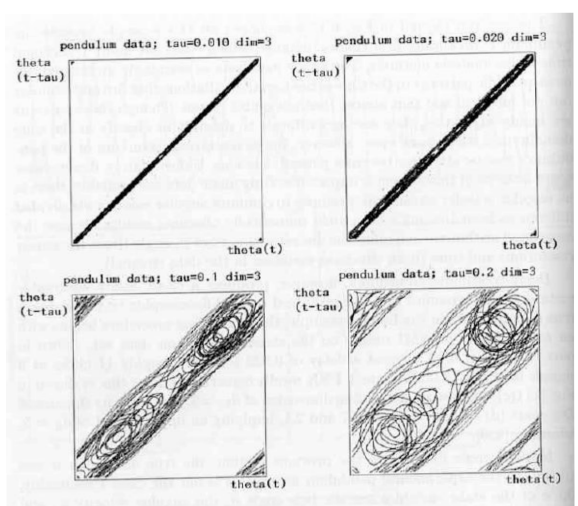
Takens* theorem:

For the right τ and enough dimensions, the embedded dynamics are diffeomorphic to (have same topology as) the original state-space dynamics.





Picking τ:



TISEAN contains tools that help you do this (e.g., `mutual`)

Picking *m*:

m > 2d: sufficient to ensure no crossings in reconstruction space:

…may be overkill.

"Embedology" paper: *m* > 2 dc
(box-counting dimension)

TISEAN contains tools that help you do this (e.g., `false_nearest`)

If Δt is not uniform:

Theorem (Takens): for τ>0 and m > 2d, reconstructed trajectory is diffeomorphic to the true trajectory

Conditions: evenly sampled in time

Interspike interval embedding:

idea: lots of systems generate spikes — hearts, nerves, etc.

if you assume that the spikes are the result of an integrate-and-fire system, then the Δt has a one-to-one correspondence to some state variable's *integrated* value…

in which case the Takens theorem still holds.

(with the Δts as state variables)

Sauer, *Chaos* **5**:127