Robustness Lab

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1 Theory

We consider the following causal diagram with the nodes U, A, B, and C:

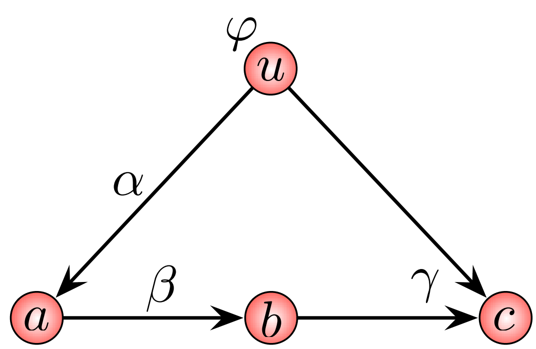

Figure 1: Causal diagram.

In this diagram, a directed edge between two nodes represents a direct causal effect. For example, the edge from U to C means that U can have a direct causal influence on C. In addition to the graph, the mechanisms φ, α, β, and γ are given. They describe how the nodes function and are formalized in terms of stochastic matrices (tiger/cat $\times$ run away/towards matrices). For example, $\gamma(u, b; c)$ stands for the probability that node C goes to state c given that it has received b and u. Based on these mechanisms, the probability of observing the states u, a, b, and c in the unperturbed system can be computed as the product

$$p(u, a, b, c) \;=\; \varphi(u) \cdot \alpha(u; a) \cdot \beta(a; b) \cdot \gamma(u, b; c)\,. \tag{1}$$

This equation connects the phenomenological level (left-hand side of equation (1)) and the mechanistic level (the individual terms on the right-hand side of equation (1)). In order to describe causal effects, say the causal effect of a on c, one has to apply an interventional operation, which Jessica Flack called *clamping*. Clamping implies a change of mechanisms, and it is formalized by the so-called *do*-operation. For example, if we clamp $\hat{a}$, or, in Pearl's terminology [5], do $\hat{a}$, we would have to replace equation (1) by

$$p_{\hat{a}}(u, a, b, c) \;=\; \varphi(u) \cdot \hat{\alpha}(a) \cdot \beta(a; b) \cdot \gamma(u, b; c)\,, \tag{2}$$

where $\hat{\alpha}(a) = 1$, if $a = \hat{a}$, and $\hat{\alpha}(a) = 0$, if $a \neq \hat{a}$. In particular, after clamping node A the new mechanism $\hat{\alpha}$ is not sensitive to u anymore. In terms of the *do*-formalism, the post-interventional probability measure $p_{\hat{a}}(u, a, b, c)$ is written as $p_{\hat{a}}(u, a, b, c) = p(u, a, b, c \,|\, do(\hat{a}))$. Summation over u, a, b yields the probability of observing c after having clamped $\hat{a}$:

$$p(c \,|\, do(\hat{a})) \;=\; \sum_{u,b} \varphi(u) \cdot \beta(\hat{a}; b) \cdot \gamma(u, b; c)\,. \tag{3}$$

We refer to these post-interventional probability measures as causal effect of A on C (see Figure 2). The equation (3) provides a formal definition of a causal effect. It can be determined in

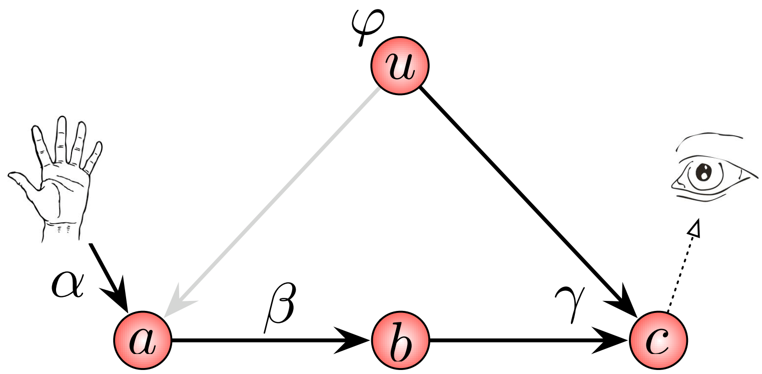

Figure 2: Intervention.

various ways, depending on the available experimental operations: If we can experimentally intervene into the system, then the mechanisms will generate the post-interventional probability measure which can be observed (see Figure 2). It turns out that, if we do not have the possibility to intervene into the system, but can observe A, B, and C, then we can still determine the causal effect of A on C. More precisely, the following formula holds:

$$p(c \,|\, do(a)) \;=\; \sum_b p(b \,|\, a) \sum_{a'} p(a') \cdot p(c \,|\, a', b) \tag{4}$$

(For simplicity, the symbol $\hat{a}$ is replaced by a.) For comleteness, I provide the proof of this equation.

Proof of (4): We use formula (3).

$$\begin{aligned}
p(c \,|\, do(a)) &= \sum_{u,b} \varphi(u) \cdot \beta(a;b) \cdot \gamma(u,b;c) \\
&= \sum_{u,b} p(u) \cdot p(b \,|\, a) \cdot p(c \,|\, u, b) \\
&= \sum_{u,b} \left(\sum_{a'} p(a') \cdot p(u \,|\, a') \right) p(b \,|\, a) \cdot p(c \,|\, u, b) \\
&= \sum_b p(b \,|\, a) \sum_{a'} p(a') \sum_u p(u \,|\, a') \cdot p(c \,|\, u, b) \\
&= \sum_b p(b \,|\, a) \sum_{a'} p(a') \sum_u p(u \,|\, a', b) \cdot p(c \,|\, u, b, a') \\
&= \sum_b p(b \,|\, a) \sum_{a'} p(a') \cdot p(c \,|\, a', b)
\end{aligned}$$

□

2 Example

Now we apply the theory to an example. In this example we assume that all nodes are binary with states 0 and 1. The following matrices specify the individual mechanisms:

$$\varphi = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad \alpha = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \qquad \beta = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \qquad \gamma = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Problem 1: Assume that for some reason you know all these mechanisms. Compute the causal effect of A on C using formula (3), which is based on mechanisms:

$$\begin{array}{ll} p(C=0 \,|\, do(A=0)) = \ldots, & p(C=1 \,|\, do(A=0)) = \ldots, \\ p(C=0 \,|\, do(A=1)) = \ldots, & p(C=1 \,|\, do(A=1)) = \ldots. \end{array} \tag{5}$$

Problem 2: Assume that you do not know the mechanisms. Instead you have observed a set of samples (a, b, c) (see the list below) that has been generated by the above mechanisms. Estimate the marginal $p(a, b, c)$ from these samples and use formula (4) to determine the casual effects (5).

001	011	011	110	110	000	000	110	100	101
101	110	011	101	110	100	100	111	001	100
000	000	000	000	001	100	100	000	001	111
001	001	000	010	011	111	100	111	111	100
100	001	001	001	001	100	000	100	110	010

Problem 3: Compute the marginal probability measure $p(a, b, c)$ using formula (1) and

$$p(a, b, c) \;=\; \sum_u p(u, a, b, c)\,.$$

More precisely,

$$\begin{array}{ll} p(A=0, B=0, C=0) = \ldots, & p(A=1, B=0, C=0) = \ldots, \\ p(A=0, B=1, C=0) = \ldots, & p(A=0, B=0, C=1) = \ldots, \\ p(A=1, B=1, C=0) = \ldots, & p(A=1, B=0, C=1) = \ldots, \\ p(A=0, B=1, C=1) = \ldots, & p(A=1, B=1, C=1) = \ldots. \end{array}$$

Problem 4: Use the marginal probability measure $p(a, b, c)$ in order to compute the causal effect of A on C using formula (4). Compare this solution with the solution of Problems 1 and 2.

References

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